2023

B.A./B.Sc.

Sixth Semester

CORE – 14

MATHEMATICS

Course Code: MAC 6.21

(Ring Theory & Linear Algebra - II)

Total Mark: 70 *Pass Mark: 28*

Time: 3 hours

Answer five questions, taking one from each unit.

UNIT–I

1. (a) If $R[x]$ is the ring of polynomials of a ring R, prove that R is an integral domain if and only if $R[x]$ is an integral domain. What can you say about $R[x]$ if R is a field? 5

   (b) State and prove the division algorithm for polynomials. 5

   (c) Let F be a field. If $f(x) \in F[x]$ and deg $f(x)$ is 2 or 3, prove that $f(x)$ is reducible over F if and only if $f(x)$ has a zero in F. 4

2. (a) Prove that the product of two primitive polynomials is primitive. 5

   (b) If $f(x) \in Z[x]$ is reducible over Q (the field of rational), prove that $f(x)$ is reducible over Z (the ring of integers). 4

   (c) If F is a field, prove that $F[x]$ is a principal ideal domain. 5

UNIT–II

3. (a) Prove that every Euclidean domain is a principal ideal domain. 5

   (b) Show that $\frac{Q[x]}{I}$, where $I = (x^2 - 5x + 6)$ is not a field. 2

   (c) Prove that every principal ideal domain is a unique factorization domain. 7

4. (a) Show that $\sqrt{-5}$ is a prime element in the ring

   $Z\left[\sqrt{-5}\right] = \left\{a + \sqrt{-5}b : a, b \in \mathbb{Z}\right\}$ 3

(b) If R is an integral domain with unity in which every non-zero, non unit element is a finite product of irreducible elements and every irreducible elements is prime, prove that R is unique factorization domain. 7

(c) Prove that every irreducible element in $R[x]$ is an irreducible polynomial in $R[x]$, R being an integral domain with unity. Is the converse true? Justify. 4

UNIT–III

5. (a) If $B = \{(1,-2,3), (1,-1,1), (2,-4,7)\}$ is a basis of $V_3(\mathbb{R})$, find its dual basis. 5

(b) If $\{v_1, v_2, ..., v_n\}$ is a basis of the vector space V over a field F and $\{f_1, f_2, ..., f_n\}$ is the dual basis of $\{v_1, v_2, ..., v_n\}$, then prove that

(i) any vector $v \in V$ is expressible as $v = \sum_{i=1}^{n} f_i(v) v_i$.

(ii) any linear functional $f \in v^*$ is expressible as $f = \sum_{i=1}^{n} f_i(v) f_i$. 5

(c) If $T : V \to U$ is a linear map and $T^* : U^* \to V^*$ is its transpose, show that the Kernel of T^* is the annihilator of the image of T. 4

6. (a) Let $\lambda_1, \lambda_2, ... \lambda_k$ be the distinct eigen values of $T \in L(V)$. If W_i be the eigen space corresponding to the eigen values λ_i and if $W = W_1 + W_2 + ... + W_k$ then prove that $\dim W = \dim W_1 + \dim W_2 + ... + \dim W_k$ 4

(b) Prove that the minimal polynomial of a matrix of linear operator is unique. 4

(c) If T is a linear operator on R^3 which is represented in the standard ordered basis by the matrix $A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$, prove that T is diagonalizable by exhibiting a basis of R^3 each of which is a eigen vector of *T*. 6

UNIT–IV

7. (a) Prove that every finite dimensional vector space is an inner product space. 5

(b) Find an orthonormal basis of the vector space *V* of all real polynomials of degree not greater than two, in which the inner product space is defined as $\langle \phi(x), \psi(x) \rangle = \int_{-1}^{1} \phi(x), \psi(x) dx$ where $\phi(x), \psi(x) \in V$. 6

(c) Prove that an orthonormal set of vectors is linearly independent. 3

8. (a) Let *W* be a subspace of an inner product space *V*. Show that

(i) $V = W \oplus W^{\perp}$ (ii) $W = W^{\perp\perp}$ 5

(b) If $\{ u_1, u_2, ..., u_n \}$ be any finite orthonormal set in an inner product space *V* over a field *F* and if *u* is any vector in *V* then prove that $\sum_{i=1}^{n} |\langle u, u_i \rangle|^2 \leq \|u\|^2$. Furthermore, show that equality holds if and only if *u* is the subspace generated by $\{u_1, u_2, ..., u_n\}$. 6

(c) Find the norm of the vector $v = (1, -2, 5)$. Also normalize this vector. 3

UNIT–V

9. (a) If *T* and *S* are linear operators on an inner product space *V* over field *F*, prove that

(i) $(T + S)^* = T^* + S^*$ (ii) $(aT)^* = \overline{a}T^*$ where $a \in F$

(ii) $(TS)^* = S^* T^*$ (iv) $(T^*)^* = T$ 4

(b) Let $A \in M_{n \times n}(F)$ and $y \in F^n$. Then prove that there exists $x_o \in F^n$ such that $(A^* A) x_o = A^* y$ and $\|Ax_0 - y\| \leq \|Ax - y\| \forall y \in F^n$. Furthermore, if $rank(A) = n$ then prove that $x_0 = (A^* A)^{-1} A^* y$. 5

(c) Suppose T is a linear operator on an inner product space $V(F)$. Then show that the adjoint T^* of T exists such that $TT^* = T^*T = I$ if and only if T is unitary. 5

10. (a) Let T be a linear operator on a complex inner product space V. Then prove that T is normal if and only if $\|T^*(u)\| = \|T(u)\| \; \forall u \in V$. 4

(b) Show that a linear operator E is a perpendicular projection in an inner product space if and only if it is idempotent and self-adjoint. 6

(c) Prove that if T be a normal operator on a finite dimensional complex inner product space V, then every subspace of V which is invariant under T is also invariant under T^*. 4