2023

B.A./B.Sc.

Sixth Semester

CORE – 13

MATHEMATICS

Course Code: MAC 6.11

(Metric Spaces & Complex Analysis)

Total Mark: 70 *Pass Mark: 28*

Time: 3 hours

Answer five questions, taking one from each unit.

UNIT–I

1. (a) Let $X = \mathbb{R}^n$ and define $d(x,y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{½}$ for all $x, y \in \mathbb{R}^n$. Show that (X, d) forms a metric space. 5

(b) If A is a subset of a metric (X, d), then show that A° is an open subset of A that contains every open subset of A. 5

(c) Let (X, d) be a metric space. Define $d' : X \times X \to \mathbb{R}$ by $d'(x,y) = \dfrac{d(x,y)}{1 + d(x,y)}, \forall x, y \in X$. Prove that d' is also a metric on X. 4

2. (a) If F is a subset of the metric space (X, d), then show that F is closed in X if and only if F^c is open in X. 5

(b) Prove that the metric space (X, d) is complete if and only if, for every nested sequence $\{F_n\}_{n \geq 1}$ of nonempty closed subsets X, the intersection $\bigcap_{n=1}^{\infty} F_n$ contains one and only one point. 5

(c) Show that a convergent sequence in a metric space is a Cauchy sequence. 4

UNIT–II

3. (a) Prove that a mapping $f : X \to Y$ is continuous on X if and only if $f^{-1}(G)$ is open in X for all open subsets G of Y. 5

(b) Let (X, d) be metric space. Then show that the following statements are equivalent: 5

(i) (X, d) is disconnected

(ii) There exist two nonempty disjoint subsets A and B, both open in X, such that $X = A \cup B$.

(iii) There exist two nonempty disjoint subsets A and B, both closed in X, such that $X = A \cup B$.

(iv) There exists a proper subset of X that is both open and closed in X.

(c) Let (X, d_X) and (Y, d_Y) be two metric spaces and $f : X \to Y$ be uniformly continuous. If $\{x_n\}_{n\geq1}$ is Cauchy in X, show that $\{f(x_n)\}_{n\geq1}$ is also Cauchy in Y. 4

4. (a) Let (X, d) be metric space and let $x \in X$ and $A \subseteq X$ be nonempty. Prove that $x \in \overline{A}$ if and only if $d(x, A) = 0$. 5

(b) Let $T : X \to X$ be a contraction of the complete metric space (X, d). Then prove that T has a unique fixed point. 5

(c) Let (X, d_X) and (Y, d_Y) be metric spaces, $\{f_n\}_{n\geq1}$ a sequence of functions, each defined on X with values in Y, and let $f : X \to Y$. Suppose that $f_n \to f$ uniformly over X and that each f_n is continuous over X. Prove that f is continuous over X. 4

UNIT–III

5. (a) Prove that when the limit of a function $f(z)$ exists at a point z_0, it is unique. 4

(b) Discuss the differentiability of the function $f(z) = |z|^2$. 5

(c) Use Cauchy-Riemann equations for polar coordinates to show that $f(z) = \frac{1}{z^4} (z \neq 0)$ is differentiable and hence compute $f'(z)$. 5

6. (a) Show that the limit of the function $f(z) = \left(\frac{z}{\overline{z}}\right)^2$ as z tends to 0 does not exist. 4

(b) If a function $f(z)$ is continuous and nonzero at the point z_0, then prove that $f(z) \neq 0$ throughout some neighbourhood of that point. 4

(c) Suppose that $f(z) = u(x, y) + iv(x, y)$, $z_0 = x_0 + iy_0$, and $w_0 = u_0 + iv_0$. Prove that $\lim_{z \to z_0} f(z) = w_0$ if and only if

$$\lim_{(x,y)\to(x_0,y_0)} u(x, y) = u_0 \text{ and } \lim_{(x,y)\to(x_0,y_0)} v(x, y) = v_0 .$$ 6

UNIT–IV

7. (a) Suppose that if a function $f(z)$ and its conjugate $\overline{f(z)}$ are both analytic in a given domain D, then show that $f(z)$ must be a constant. 5

(b) Evaluate $\int_C \frac{z+2}{z} dz$ where C is the semi circle $z = 2e^{i\theta} (\pi \le \theta \le 2\pi)$. 5

(c) Show that e^z is entire. 4

8. (a) Evaluate the following: 2×2=4

(i) $\log(-1 - i\sqrt{3})$

(ii) $(1 + i)^i$

(b) Without evaluating the integral, show that $\left| \int_C \frac{dz}{z^2 - 1} \right| \le \frac{\pi}{3}$, where C is the arc of the circle $|z| = 2$ from $z = 2$ to $z = 2i$. 4

(c) State and prove the Cauchy integral formula. 6

UNIT–V

9. (a) State and prove the fundamental theorem of algebra, using Liouville's theorem. 5

(b) If a power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges when $z = z_1 (z_1 \neq z_0)$, then show that it is absolutely convergent at each point z in the open disc $|z - z_0| < R_1$, where $R_1 = |z_1 - z_0|$. 5

(c) Represent the function $f(z)=\dfrac{z}{(z-1)(z-3)}$ by a series of negative powers of $(z-1)$ which converges to $f(z)$, when $0<|z-1|<2$. 4

10. (a) Suppose that $z_n = x_n + iy_n (n = 1, 2, ...)$ and $S = X + iY$, then prove that $\sum_{n=1}^{\infty} z_n = S$ if and only if $\sum_{n=1}^{\infty} x_n = X$ and $\sum_{n=1}^{\infty} y_n = Y$. 5

(b) If z_1 is a point inside the circle of convergence $|z-z_0|=R$ of a power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$, then prove that this series must be uniformly convergent in the closed disk $|z-z_0| \le R_1$, where $R_1 = |z_1 - z_0|$. 5

(c) Represent the function $f(z)=\dfrac{4z+3}{z(z-3)(z+2)}$ in Laurent's series in the annular region between $|z|=2$ and $|z|=3$. 4