**2023**

**B.A./B.Sc.**

**Fourth Semester**

CORE – 10

**MATHEMATICS**

*Course Code: MAC 4.31*

(Ring Theory & Linear Algebra - I)

*Total Mark: 70* *Pass Mark: 28*

*Time: 3 hours*

*Answer five questions, taking one from each unit.*

## UNIT–I

1. (a) Define integral domain and field and prove that every field is an integral domain. What can you say about the converse? 5

   (b) Prove that the set $M_2(\mathbb{Z})$ of 2×2 matrices with entries in $\mathbb{Z}$ is a ring under matrix addition and matrix multiplication. 5

   (c) Show that a ring R is commutative if and only if $(a+b)^2 = a^2 + b^2 + 2ab$, for all $a, b \in R$. 4

2. (a) Define subring of a ring and show that the sum of two subrings of a ring *R* may not be a subring of *R*. 3

   (b) Define characteristic of a ring and prove that the characteristic of an integral domain is either zero or a prime number. 5

   (c) Give an example to show that the union of two subrings of a ring *R* may not be a subring of *R* and prove that the union of two subrings of a ring *R* is a subring of *R* if and only if one of them is contained in the other. 6

## UNIT–II

3. (a) Define ideal of a ring and prove that every ideal of a ring *R* is a subring of *R*. Is the converse true? Justify. 4

   (b) Define maximal ideal of a ring and prove that an ideal *M* of a commutative ring *R* with unity is a maximal ideal of *R* if and only if $R/M$ is a field. 7

(c) If $J$ is an ideal of a ring $R$ with unity such that $1 \in J$, then prove that $J = R$. 3

4. (a) If $I$ and $J$ are two ideals of a ring $R$, then prove that $1 + J = \langle I \cup J \rangle$. 5

(b) Define prime ideal of a ring and prove that an ideal P of a commutative ring $R$ is a prime ideal of $R$ if and only if $R/P$ is an integral domain. 6

(c) If $F$ is a field, prove that its only ideals are $\{0\}$ and $F$. 3

## UNIT–III

5. (a) Define ring isomorphism. If $R$ and $R'$ are two rings and $f : R \to R'$ is an onto homomorphism, then prove that $R' \cong R / \ker f$, where $\ker f$ denotes the kernel of $f$. 6

(b) If $\mathbb{C}$ denotes the field of complex numbers, show that the map $f : \mathbb{C} \to \mathbb{C}$ defined by $f(x+iy) = x - iy$, is a ring homomorphism. 4

(c) Prove that any homomorphism of a field is either injective or takes each element to zero. 4

6. (a) If $R$ and $R'$ are two rings and $f : R \to R'$ is a homomorphism, define kernel of $f$ and prove that it is an ideal of $R$. 3

(b) If $R$ and $R'$ are two rings and $f : R \to R'$ is a homomorphism, prove that

(i) $f(0) = 0'$ where 0 and 0' denote the zero elements of $R$ and $R'$ respectively.

(ii) $f(-a) = -f(a)$ for all $a \in R$.

3

(c) Define embedding and prove that an integral domain can be embedded in a field. 8

## UNIT–IV

7. (a) Prove that every linearly independent subset of a finitely generated vector space can be extended to form a basis. 5

(b) If $X$ and $Y$ are subspaces of a finite dimensional vector space, prove that $\dim(X+Y) = \dim X + \dim Y - \dim(X \cap Y)$. 6

(c) Let $V$ be the vector space of functions from $\mathbb{R}$ to $\mathbb{R}$. Show that $f, g, h \in V$ are linearly independent, where $f(x) = e^{2x}, g(x) = x^2, h(x) = x$. 3

8. (a) Define linear span and prove that the linear span of any non–empty subset of a vector space is a subspace of the vector space. 5

(b) If $V$ is a finite dimensional vector space over a field $F$ with dimension $n$, prove that any set with $(n+1)$ or more vectors of $V$, is linearly dependent. 5

(c) If the vectors $(0, 1, a)$, $(1, a, 1)$, $(a, 1, 0)$ of the vector space $\mathbb{R}^3(\mathbb{R})$ are linearly dependent, then find the value of $a$. 4

## UNIT–V

9. (a) Define rank and nullity of a linear transformation. State and prove the rank-nullity theorem. 8

(b) If the matrix representation of a linear transformation $T$ on the vector space $\mathbb{R}^3(\mathbb{R})$ with respect to the basis $\{(1,0,0), (0,1,0), (0,0,1)\}$ is

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix},$$

what is the matrix representation of $T$ with respect to the basis $\{(0, 1, -1), (1, -1, 1), (-1, 1, 0)\}$? 6

10. (a) If $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a map given by $T(x, y, z) = (x, y) \forall (x, y, z) \in \mathbb{R}^3$, show that $T$ is a linear transformation and also find the null space of $T$. 4

(b) State and prove the first isomorphism theorem of linear algebra. 6

(c) If $T$ is a linear transformation on a vector space $V$ over a field $F$ such that $T^2 - T + 1 = 0$, then show that $T$ is invertible. 4

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