2023

B.A./B.Sc.

Fourth Semester

CORE – 9

MATHEMATICS

Course Code: MAC 4.21

(Riemann Integration & Series of Functions)

Total Mark: 70 *Pass Mark: 28*

Time: 3 hours

Answer five questions, taking one from each unit.

UNIT–I

1. (a) Let $f:[a,b]\to\mathbb{R}$ be bounded and $c\in(a,b)$. Let P_1 and P_2 be partitions of $[a, c]$ and $[c, b]$ respectively. Show that $P=P_1\cup P_2$ is a partition of $[a, b]$. Also, show that $L(f,P)=L(f,P_1)+L(f,P_2)$ and $U(f,P)=U(f,P_1)+U(f,P_2)$. 4

   (b) Let $f:[a,b]\to\mathbb{R}$ be bounded such that f^2 is integrable over $[a, b]$, where $f^2(x)=\{f(x)\}^2$. Then f is integrable over $[a, b]$. Prove or disprove. 4

   (c) Let $f:[-1,1]\to\mathbb{R}$ be defined by $f(x)=\begin{cases} a, & \text{if } -1\le x<0 \\ 0, & \text{if } x=0 \\ b, & \text{if } 0<x\le 1 \end{cases}$

   where $b<0<a$. Is f integrable over $[-1, 1]$? Justify. 6

2. (a) Let $f:[a,b]\to\mathbb{R}$ be defined by $f(x)=\begin{cases} k, & \text{if } x \text{ is rational} \\ -k, & \text{if } x \text{ is irraxtional} \end{cases}$

   where $k\ne 0$. Is f integrable over $[a,b]$? Justify. 4

   (b) Let $f:[a,b]\to\mathbb{R}$ be bounded and continuous. If $\int_a^b f(x)dx=0$, prove that $f(c)=0$ for at least one $c\in[a,b]$. 4

(c) Let $f:(0,1)\to\mathbb{R}$ be defined by $f(x)=\begin{cases}0, & \text{if } x \text{ is irrational}\\ \dfrac{1}{q}, & \text{if } x=\dfrac{p}{q}\end{cases}$

where $p, q \in N$ having no common factor. Show that f is integrable (0,1) and hence, evaluate $\int_0^1 f(x)dx$. 6

UNIT–II

3. (a) Let $f:[a,b]\to\mathbb{R}$ be integrable over $[a, b]$ and let $c\in\mathbb{R}$. Prove that cf is integrable over $[a, b]$ and $\int_a^b (cf)(x)dx = c\int_a^b f(x)dx$. 7

(b) Let $h:[a,b]\to\mathbb{R}$ be integrable over $[a, b]$. If $h(x)\geq 0$ for all $x\in[a,b]$, prove that $\int_a^b h(x)dx \geq 0$.

Hence, prove the following:

Let $f, g:[a,b]\to\mathbb{R}$ be integrable over $[a, b]$. If $f(x)\geq g(x)$ for all $x\in[a,b]$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$. 5+2=7

4. (a) Let $f:[a,b]\to\mathbb{R}$ be bounded and monotone. Prove that f is integrable over $[a, b]$. 7

(b) Let $f:[a,b]\to\mathbb{R}$ be bounded. Suppose that f is continuous in $[a, b]$ except at one point $c\in(a,b)$. Show that f is integrable over $[a, b]$. Hence, prove that f is integrable over $[a, b]$ if f has only finitely many points of discontinuity in $[a, b]$. 4+3=7

UNIT–III

5. (a) For what values of $p\in\mathbb{R}$ does the integral $\int_1^\infty \frac{1}{x^p}dx$ converge? Justify. 5

(b) Prove that $\Gamma(a+1)=\Gamma(a)$ for $a>0$.
Deduce that $\Gamma(n+1)=n!$ for $n\in\mathbb{N}$. 4+1=5

(c) Examine the convergence of $\int_0^1 \frac{dx}{x^{\frac{1}{3}}(1+x^2)}$ 4

6. (a) Show that $\int_0^\infty x^{t-1}e^{-x}dx$ is convergent for $t>0$. 5

(b) For $a,b>0$, show that $B(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$. 5

(c) Examine the convergence of $\int_0^1 \frac{dx}{x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}}$ 4

UNIT–IV

7. (a) Consider the sequence of functions $f_n:[0,\infty)\to\mathbb{R}$ defined by $f_n(x)=\frac{x^n}{n+x^n}$ and $f:[0,\infty)\to\mathbb{R}$ defined by $f(x)=\begin{cases}0, & \text{if } 0\le x\le 1\\ 1, & \text{if } x>1\end{cases}$. Prove that $f_n\to f$ pointwise on $[0,\infty]$. Does $f_n\to f$ uniformly on $[0,\infty)$? Justify.
5+1=6

(b) Let $f:\mathbb{R}\to\mathbb{R}$ be uniformly continuous on $\mathbb{R}$. Consider the sequence of functions $f_n(x)=f\left(x+\frac{1}{n}\right)$. Show that $f_n\to f$ uniformly on $\mathbb{R}$. 4

(c) Let $a>0$. Prove that the series of functions $\sum_{n=1}^{\infty}\frac{x}{n(1+nx^2)}$ is uniformly convergent on any interval $[a,b]$. 4

8. (a) Let $X\subseteq\mathbb{R}$ and let $f_n:X\to\mathbb{R}$. Prove that (f_n) is uniformly convergent on *X* if and only if (f_n) is uniformly Cauchy on *X*. 7

(b) Let $f_n, f : [a,b] \to \mathbb{R}$ such that $\sum_{n=1}^{\infty} f_n = f$ uniformly on $[a, b]$. If each f_n is integrable over $[a, b]$, prove that f is integrable over $[a, b]$ and $\int_a^b f(x)dx = \sum_{n-1}^{\infty} \int_a^b f_n(x)dx$. 7

UNIT–V

9. (a) Let (a_n) be a bounded sequence of real numbers and let $L = \limsup a_n$. Prove that 4+3=7

(i) there exists $N \in \mathbb{N}$ such that $a_n < L + \varepsilon$ for all $n \geq \mathbb{N}$

(ii) $L - \varepsilon < a_n$ for infinitely many n

(b) Suppose that the coefficients of the power series $\sum_{n=0}^{\infty} a_n x^n$ are all integers and infinitely many of them are non-zero. Prove that the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$ is at most 1. 3

(c) Determine the radius of convergence of the following power series:

(i) $\sum_{n=0}^{\infty} \frac{2^n}{n!} x^n$ (ii) $\sum_{n=0}^{\infty} \frac{n^3}{3^n} x^n$ 2+2=4

10. (a) Let $\sum_{n=0}^{\infty} a_n (x-a)^n$ be a power series whose coefficients $a_n \neq 0$ for all $n \geq 0$. If $\lim_{x\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists, prove that the limit equals $\lim_{x\to\infty} |a_n|^{\frac{1}{n}}$. 7

(b) For $x \in \mathbb{R}$, prove that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. 3

(c) Let $\mathbb{R}$ be the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$. What can you say about the convergence of $\sum_{n=0}^{\infty} a_n x^n$ at $x = \mathbb{R}$? Justify. 4