2023

B.A./B.Sc.

Second Semester

CORE – 4

MATHEMATICS

Course Code: MAC 2.21

(Differential Equations)

Total Mark: 70 *Pass Mark: 28*

Time: 3 hours

Answer five questions, taking one from each unit.

UNIT–I

1. (a) Solve the initial value problem

   $(2xy-3)\,dx+(x^2+4y)\,dy=0,\ \ y(1)=2$ 5

   (b) Solve the equation $\left(y+\sqrt{x^2+y^2}\right)dx-xdy=0$ 5

   (c) Show that first-order differential equation $\left(\dfrac{dy}{dx}\right)^2-4y=0$ has a one-parameter family of solutions of the form $f(x)=(x+c)^2$, where c is an arbitrary constant. 4

2. (a) Solve : $(5x+2y+1)dx+(2x+y+1)dy=0$ 5

   (b) Solve: $(y+2)dx+y(x+4)dy=0$ 5

   (c) Determine the most general function $N(x,y)$ such that the given equation is exact $(x^3+xy^2)dx=N(x,y)dy=0$ 4

UNIT–II

3. (a) In a lake, the pollution level is 5%. If the fresh water at the rate of 10000 litres per day is allowed to enter and the same amount of water leaves the lake, find the time when the pollution level is 2.5% if volume of lake is 500000 litres. Further, if safety level is 0.1%, then after how much time will the water be suitable for drinking?

   3+2=5

(b) The mass of a radioactive isotope decays at a rate proportional to the mass of the isotope present. If the half life of the isotope is 80 years, determine the percentage of the original amount which remains after 50 years. 4

(c) The initial population of a region is 100. Suppose the population can be modelled using the differential equation $\frac{dx}{dt} = 0.2x - 0.001x^2$ with a time step of one month. Find predicted population after 2 months. 5

4. (a) The initial amount of a radioactive nuclei is 6.00×10^{24}. 10 days later this number is reduced to 6.25×10^{22}. If $\frac{dN}{dt} = -kN$ is the rate of decay of the nuclei (where N is the number of nuclei present at time t).

   (i) Show that $k = 0.45643$, correct to five decimal places. 3

   (ii) Calculate the number of the radioactive nuclei after a further period of 10 days has lapsed. 2

   (b) Out of 20,000 chickens in a farm, some chickens were infected by a virus. The rate at which the chickens are infected is proportional to the product of the number of chickens infected and the number of chickens not infected.

   (i) Let t be the time in hours since the infection was first discovered. Form a differential equation for the rate at which the chickens are getting infected. 2

   (ii) Solve the differential equation for the number of infected chickens at time t given that when the disease was first discovered 4,000 chickens were infected, and chickens were infected at the rate of 32 chickens per hour. 4

   (iii) Also, find how many extra chickens will be infected after 24 hours. 3

UNIT–III

5. (a) Show that $y_1 = 1$ and $y_2 = \sqrt{x}$ are solutions of $yy'' + (y')^2 = 0$, but that their sum $y = y_1 + y_2$ is not a solution. 5

(b) Show that $y = \frac{1}{x}$ is a solution of $y' + y^2 = 0$, but if $c \neq 0$ and $c \neq 1$, then $y = \frac{c}{x}$ is not a solution of $y' + y^2 = 0$. 4

(c) Find a particular solution of the differential equation $y'' - 3y' + 2y = 0$ satisfying the initial conditions $y(0) = 1$ and $y'(0) = 0$. 5

6. (a) Let y_1 and y_2 be two solutions of $A(x)y'' + B(x)y' + C(x)y = 0$ on an open interval I where $A(x)$, $B(x)$ and $C(x)$ are continuous and $A(x)$ is never zero. Let $W = W(y_1, y_2)$. Show that

(i) $A(x)\frac{dW}{dx} = -B(x)W(x)$ 4

(ii) Sove this first-order equation to deduce Abel's formula

$$W(x) = K \exp\left(-\frac{B(x)}{A(x)}dx\right)$$ 4

(iii) Why does Abel's formula imply that $W(y_1, y_2)$ is either zero everywhere or nonzero everywhere? 1

(b) Show that $y_1 = e^x \sin x$ and $y_2 = e^x \cos x$ are linearly independent solutions of $y'' - 2y' = 2y = 0$. Also, find a particular solution which satisfies the conditions $y(0) = 0$ and $y'(0) = 5$. 5

UNIT–IV

7. (a) Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + y = \cot x$$ 5

(b) Solve the initial value problem

$$\left(D^2 - 3D + 4\right)y = 9x^2 + 4,\ y(0) = 6, y'(0) = 8$$

D stands for $\frac{d}{dx}$ 5

(c) Solve $\left(4D^2+12D+9\right)y=144e^{-3x}$ using the method of undetermined coefficients. 4

8. (a) Solve $\dfrac{d^3y}{dx^3}-6\dfrac{d^2y}{dx^2}+11\dfrac{dy}{dx}-6y=e^x$ 6

(b) Solve $(D^2+1)y=\tan x$ using method of variation of parameters. 5

(c) Find the general solution of $x^2\dfrac{d^2y}{dx^2}-6x\dfrac{dy}{dx}+10y=3x^4+6x^3$ given that $y=x^2$ and $y=x^5$ are linearly independent solutions of the corresponding homogeneous equation. 3

UNIT–V

9. (a) Consider a disease where all those infected remain contagious for life. A model describing this is given by the differential equations $\dfrac{dS}{dt}=-\beta SI$ and $\dfrac{dI}{dt}=\beta SI$ (the symbols have their usual meanings), where β is a positive constant.

   (i) Use chain rule to find a relation between S and I. 5

   (ii) Obtain and sketch the phase plane curves. Determine the direction of travel along the trajectories. 4

(b) The following battle model represents two armies where both are exposed to aimed fire: $\dfrac{dR(t)}{dt}=-a_1B(t)$, $\dfrac{dB(t)}{dt}=-a_2R(t)$

where $R(t)$ and $B(t)$ denote the number of soldiers in the red army and blue army respectively and a_1 and a_2 are the attrition coefficients (positive).

Let us suppose that both armies have equal attrition coefficients and that red army has 10,000 soldiers initially and the blue army has 8,000 soldiers. Determine who wins if

   (i) there is one battle between the two armies. 2

(ii) there are two battles: the first half with half the red army against the entire blue army and the second with the other half of the red army against the blue army survivors of the first battle. 3

10. (a) A population of sterile rabbits $X(t)$ is preyed upon by a population of foxes $Y(t)$. A model for this population interaction is the pair of differential equations $\frac{dX}{dt} = -aXY$, $\frac{dY}{dt} = bXY - cY$ where, a, b and c are positive constants.

(i) Use chain rule to obtain a relationship between the density of foxes and the density of rabbits. 5

(ii) Sketch typical phase-plane trajectories, indicating the direction of movement along the trajectories. 4

(iii) According to this model, is it possible for the foxes to completely wipe out the rabbit population? Give reasons. 2

(b) Find all equilibrium solutions of

$$\frac{dx}{dt} = 14x - 2x^2 - xy, \quad \frac{dy}{dt} = 16y - 2y^2 - xy$$ 3