2023
B.Sc.
Second Semester
CORE – 3
MATHEMATICS
Course Code: MAC 2.11
(Real Analysis)

Total Mark: 70 *Pass Mark: 28*
Time: 3 hours

Answer five questions, taking one from each unit.

UNIT–I

1. (a) For $a \neq 0$ and b in R, prove that 4
   (i) if $a.b = 1$, then $b = 1/a$
   (ii) if $a.b = 0$, then $a = 0$ or $b = 0$
   (b) Prove that a set N is a neighbourhood of the point p if and only if there exists a positive integer n such that $\left(p - \frac{1}{n}, p + \frac{1}{n}\right) \subset N$. 5
   (c) Prove that the set of all rational numbers is countable. 5

2. (a) Prove that there does not exist a rational number r such that $r^2 = 2$. 5
   (b) Prove that every infinite set has a denumerable subset. 5
   (c) Show that the set $S = \left\{(-1)^n \frac{1}{n} : n \in N\right\}$ is bounded. 4

UNIT–II

3. (a) Prove that between any two distinct real numbers there always lies a rational number and therefore infinitely many rational numbers. 5
   (b) Show that $Sup\{r \in Q : r < a\} = a$, for each $a \in R$. 4
   (c) Find the set of all the limit points of (a, b) where $a, b \in R$. Also find the derived set for these interval. 5

4. (a) Show that $|a+b| = |a| + |b|$ if and only if $ab \geq 0$ 4

(b) Prove that every bounded and infinite set has a limit point. 6

(c) Find the limit points of the following sets 4

(i) $S = \left\{ \frac{n}{n+1} : n \in N \right\}$

(ii) $(0,1)$

UNIT–III

5. (a) Prove that every bounded sequence has the greatest and the least limit point. 5

(b) Using the definition of the limit of a sequence, show that $\langle \sqrt{n^2+1} - n \rangle$ is a null sequence. 4

(c) Show that the sequence $\langle f_n \rangle$ where $f_n = 1 + \frac{1}{3} + \frac{1}{3^2} + \ldots + \frac{1}{3^{n-1}}$ converges. Also find $\lim_{x \to \infty} f_n$. 5

6. (a) Prove that the limit of a sequence if it exists is unique. 4

(b) Show that the sequence $\langle f_n \rangle$ defined by $f_n = \left(\sqrt{x^2+1} - \sqrt{x} \right) \forall x \in N$ is convergent. 5

(c) Prove that the sequence $\langle f_n \rangle$ where $f_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!}$ is monotonically increasing, bounded and tends to limit $\frac{2}{3}$. 5

UNIT–IV

7. (a) Prove that every subsequence of a convergent sequence converges. Does the converse of the statement hold? Justify. 6

(b) Show that the sequence $\langle \frac{n}{n+1} \rangle$ is a Cauchy sequence. 4

(c) Prove that if $\langle f_n \rangle$ diverges to infinity and $\langle \varphi_n \rangle$ is bounded then $\langle f_n + \varphi_n \rangle$ diverges to infinity. 4

8. (a) Using Cauchy's general principal of convergence show that the sequence $\langle f_n \rangle$ where $f_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n}$ is not convergent. 5

(b) Prove that every Cauchy sequence is bounded. Does the converse of the statement hold? Justify. 5

(c) Show that the sequence $\langle 3^n \rangle$ diverges to ∞. 4

UNIT–V

9. (a) Prove that if the series $\sum_{n=1}^{\infty} u_n$ converges, then $\lim_{n\to\infty} u_n = 0$. 4

(b) Test for convergence the infinite series $\sum \frac{2^{n-1}}{3^n + 1}$. 5

(c) Show that the series $\left(\frac{2^2}{1^2} - \frac{2}{1}\right)^{-1} + \left(\frac{3^3}{2^3} - \frac{3}{2}\right)^{-2} + \left(\frac{4^4}{3^4} - \frac{4}{3}\right)^{-3} + \ldots$ is convergent. 5

10. (a) Apply Cauchy's integral test to show that $\sum_{n=1}^{\infty} \frac{1}{n^p}, p > 0$ is convergent if $p > 1$ and divergent if $p \le 1$. 5

(b) Prove that every absolutely convergent series is convergent. 4

(c) Test for convergence, absolute convergence, and conditional convergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\log(n+1)}$. 5