April 2025
B.A./B.Sc.
Sixth Semester
DISCIPLINE SPECIFIC ELECTIVE – 4
MATHEMATICS
Course Code: MAD 6.21
(Differential Geometry)

Total Mark: 70 *Pass Mark: 28*
Time: 3 hours

Answer five *questions, taking* one *from each unit.*

UNIT–I

1. (a) Find the equation of the plane that has three points of contact at the origin to the curve $\vec{r} = \left(t^4 - 1, t^3 - 1, t^2 - 1\right)$. 5

   (b) Define involute and evolute. Find the involute of a circular helix $\vec{r} = \left(a\cos u, a\sin u, bu\right)$. 5

   (c) If the tangent and the binormal at a point of a curve make angles θ and ϕ respectively with a fixed direction, show that $\frac{\sin\theta}{\sin\phi}\cdot\frac{d\theta}{d\phi} = -\frac{\kappa}{\tau}$. 4

2. (a) Show that the curvature κ and torsion τ can be expressed as $\kappa = \left|\vec{r}\,'\times\vec{r}\,''\right|$ and $\tau = \frac{\left[\vec{r}\,', \vec{r}\,'', \vec{r}\,'''\right]}{\left|\vec{r}\,'\times\vec{r}\,''\right|^2}$. 5

   (b) For the curve $\vec{r} = \left(3t, 3t^2, 2t^3\right)$, show that $\rho = -\sigma = \frac{3}{2}\left(1 + 2t^2\right)^2$. 5

   (c) Find the equation of the osculating plane at any point to the curve $x = a\cos 2u,\ y = a\sin 2u,\ z = 2a\sin u$. 4

UNIT–II

3. (a) Define metric. Show that the metric is invariant under a transformation of parameters. 5

(b) Define direction coefficients. Find the angle between two directions on a surface at a point P having direction coefficients (l, m) and (l', m'). 5

(c) Define lines of curvature. State and prove Euler's theorem. 4

4. (a) Obtain all the fundamental magnitudes for the surface $\vec{r} = [u\cos v, u\sin v, f(v)]$. 5

(b) Show that if L, M, N vanish everywhere on a surface, then the surface is a part of a plane. 5

(c) Prove that a necessary and sufficient condition for a surface to be developable is that its Gaussian curvature should be zero. 4

UNIT–III

5. (a) Show that a necessary and sufficient condition for a curve

v = constant to be geodesic on the general surface is

$EE_2 + FE_1 - 2EF_1 = 0$. 5

(b) Discuss the nature of geodesic on the right helicoids

$x = u\cos v,\ y = u\sin v,\ z = av$. 5

(c) Prove that torsion of a geodesic vanishes in a principal direction. 4

6. (a) Show that a geodesic is either a plane curve or a line of curvature, or it is both. 5

(b) Evaluate geodesic curvature of a parametric curve v = constant. 5

(c) Prove that the torsions of two orthogonal geodesics are equal in magnitude but opposite in sign. 4

UNIT–IV

7. (a) Prove that an anti-symmetric tensor A^{ij} has $\frac{n}{2}(n-1)$

independent components. 5

(b) Prove that transformations of a covariant vector is transitive. 5

(c) Show that contraction of a mixed tensor A_j^i is a scalar invariant. 4

8. (a) If a_{ij} is symmetric tensor and b_i is a vector and $a_{ij}b_k + a_{jk}b_i + a_{ki}b_j = 0$, then show that $a_{ij} = 0$ or $b_k = 0$. 5

(b) State quotient law of tensors and use it to prove that Kronecker delta δ_j^i is a mixed tensor of rank two. 5

(c) Prove that $\frac{\partial A_i}{\partial x^j}$ is not a tensor though A_i is a tensor. 4

UNIT–V

9. (a) Show that covariant differentiation of sum and product of two tensors obey the same rules as ordinary differentiation. 5

(b) Prove that a necessary and sufficient condition for all Christoffel symbols to vanish at a point is that g_{ij} are constants. 5

(c) Prove that $[ij,k]+[jk,i]=\frac{\partial g_{ik}}{\partial x^j}$. 4

10. (a) Prove that $\left\{ \begin{matrix} i \\ ij \end{matrix} \right\} = \frac{\partial}{\partial x^j}\left(\log\sqrt{g}\right)$ where $g = \left|g_{ij}\right| \neq 0$. 5

(b) Show that covariant derivative of a covariant vector is a covariant tensor of second order. 5

(c) If ϕ be scalar function of coordinates x^i and $\vec{A}$ be an arbitrary vector, then establish that $\text{div}\left(\phi\vec{A}\right) = \phi\,\text{div}\vec{A} + \vec{A}.\vec{\nabla}\phi$. 4