April 2025
B.A./B.Sc.
Sixth Semester
CORE – 14
MATHEMATICS
Course Code: MAC 6.21
(Ring Theory & Linear Algebra - II)

Total Mark: 70 *Pass Mark: 28*
Time: 3 hours

Answer five questions, taking one from each unit.

UNIT–I

1. (a) If $R[x]$ is the ring of polynomials over a ring R, prove that $R[x]$ is commutative if and only if R is commutative. If R has unity, does $R[x]$ also have unity? Justify. 5
   (b) If F is a field such that $f(x) \in F[x]$ and deg $f(x)$ is 2 or 3, prove that $f(x)$ is reducible over F if and only if $f(x)$ has a zero in F. 5
   (c) State and prove the factor theorem for polynomials. 4

2. (a) State and prove the division algorithm for polynomials. 6
   (b) Define primitive polynomial and show that the product of two primitive polynomials is also a primitive polynomial. 5
   (c) Show that $x^2 + x + 4$ is irreducible over Z_{11}. 3

UNIT–II

3. (a) Show that every ideal of a Euclidean domain is a principal ideal. 5
   (b) Prove that in a unique factorization domain (UFD), an element is prime if and only if it is irreducible. 6
   (c) In an integral domain, show that the product of an irreducible element and a unit is an irreducible element. 3

4. (a) Prove that any two non-zero elements in a Euclidean domain have a g.c.d. 6

(b) Discuss unique factorization in $Z[x]$, where Z is the ring of integers. 8

UNIT–III

5. (a) If $\{v_1, v_2, ..., v_n\}$ is a basis of a vector space V over a field F and $f_1, f_2, ..., f_n$ are linear functionals defined by $f_i(v_j) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$

Show that $\{f_1, f_2, ..., f_n\}$ is a basis of V^*, the dual space of V. 5

(b) Define annihilator of a subset W of a vector space V over a field F and prove that it is a subspace of V^*, the dual space of V. 3

(c) State and prove Cayley-Hamilton theorem. 6

6. (a) Show that distinct non-zero eigen vectors belonging to distinct eigen values of a linear operator are linearly independent. 4

(b) If T is a linear operator on $\mathbb{R}^3$ which is represented in the standard ordered basis by the matrix $A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$, prove that T is diagonalizable by exhibiting a basis of $\mathbb{R}^3$, each of which is an eigen vector of T. 6

(c) Show that the roots of the minimal polynomial and the characteristic polynomial of a linear operator $T \in L(V)$ are same, except for their multiplicities. 4

UNIT–IV

7. (a) State and prove the triangle inequality for inner product space. 4

(b) Apply the Gram-Schmidt process to the vectors (1, 0, 1), (1, 2, 2) and (2, –1, 1) to obtain an orthonormal basis for the vector space $\mathbb{R}^3$ with the standard inner product. 6

(c) If W is a subspace of an inner product space V over a field F, show that $x \in W^{\perp}$ if and only if $\langle x, v_i \rangle = 0$ for all $i = 1, 2, 3, \ldots, n$, where $\{v_1, v_2, ..., v_n\}$ is a basis of W and $W^{\perp}$ is the orthogonal complement of W. 4

8. (a) Prove that every finite dimensional inner product space has an orthonormal basis. 7

(b) Verify whether every orthonormal set of non-zero vectors of an inner product space is linearly independent or not. 4

(c) If a linear operator T on $\mathbb{R}^2$ is defined by $T(x, y) = (x + 2y, x - y)$, find the adjoint T^* of T with the standard inner product. 3

UNIT–V

9. (a) Prove that every linear operator on a complex inner product space can be expressed as $T = T_1 + iT_2$, where T_1 and T_2 are self-adjoint. 4

(b) If T is a self-adjoint operator on an inner product space V, show that every eigen value of T is real and that eigen vectors of T corresponding to distinct eigen values are orthogonal. 5

(c) If T and S are linear operators on an inner product space V over a field F, show that $(T + S)^* = T^* + S^*$ and $(TS)^* = S^* T^*$. 5

10. (a) If T is a linear operator on a finite dimensional complex inner product space V, prove that T is normal if and only there exists an orthonormal basis for V consisting of eigen vectors of T. 5

(b) State and prove Schur's triangularization theorem. 6

(c) If T is a normal operator on an inner product space V, show that if v is an eigen vector of T, then v is also an eigen vector of T^*. 3