

April 2025
B.A./B.Sc.
Sixth Semester
CORE – 13
MATHEMATICS
Course Code: MAC 6.11
(Metric Spaces & Complex Analysis)

Total Mark: 70

Pass Mark: 28

Time: 3 hours

Answer five questions, taking one from each unit.

UNIT-I

1. (a) Let $X = \mathbb{R}^n$. For $x, y \in \mathbb{R}^n$, define $d_p(x, y) = \left(\sum_{i=1}^n |x_i - y_i|^p \right)^{1/p}$,
where $p \geq 1$. Show that (X, d_p) forms a metric space. 5
- (b) If A is a subset of a metric space (X, d) , then show that A is open if and only if $A = A^\circ$ where A° is the interior of A . 5
- (c) Let $S(x, r)$ be an open ball in a metric space (X, d) and A be a closed subset of X such that $d(A) \leq r$ and $A \cap S(x, r) \neq \emptyset$.
Show that $A \subseteq S(x, 2r)$. 4
2. (a) If a Cauchy sequence in a metric space (X, d) contains a convergent subsequence, then show that the sequence converges to the same limit as the subsequence. 5
- (b) If A is a subset of a metric space (X, d) , then show that A is closed if and only if $A = \bar{A}$. 5
- (c) Define the diameter of a set. Show that $d(A) = d(\bar{A})$ for any subset A of a metric space (X, d) . 4

UNIT-II

3. (a) If a metric space (X, d) is disconnected, show that there exists a continuous mapping of (X, d) onto the discrete two element space (X_0, d_0) . 5
- (b) If f and g are two uniformly continuous mappings of metric spaces (X, d_X) to (Y, d_Y) , and (Y, d_Y) to (Z, d_Z) , respectively, then prove that $g \circ f$ is a uniformly continuous mapping of (X, d_X) to (Z, d_Z) . 5
- (c) Let $f: \mathbb{R} \rightarrow \mathbb{C}$ be defined by $f(x) = x + ix^2$. Is f continuous at $2 \in \mathbb{R}$? Justify. 4
4. (a) Let (X, d_X) and (Y, d_Y) be metric spaces and $A \subseteq X$. Show that a function $f: A \rightarrow Y$ is continuous at $a \in A$ if and only if whenever a sequence $\{x_n\}$ in A converges to a , the sequence $\{f(x_n)\}$ converges to $f(a)$. 6
- (b) Let (X, d) and (X, ρ) be metric spaces, such that
$$\rho(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$
 for all $x, y \in X$. Show that d and ρ are equivalent. 4
- (c) Let A be a subset of the metric space (X, d) . Define $f(x) = d(x, A) = \inf \{d(x, y) : y \in A\}$, $x \in X$. Show that f is uniformly continuous over X . 4

UNIT-III

5. (a) Prove that the function $f(z) = |z|^2$ is continuous everywhere but not differentiable except at the origin. 5
- (b) If z_0 and w_0 are points in the z and w planes, respectively, then prove that $\lim_{z \rightarrow z_0} f(z) = \infty$ if and only if $\lim_{z \rightarrow z_0} \frac{1}{f(z)} = 0$. 5

- (c) Prove or disprove that, when a limit of a complex-valued function $f(z)$ exists at a point z_0 , then it must be unique. 4
6. (a) Show that $\lim_{(x,y) \rightarrow (x_0,y_0)} u(x,y) = u_0$ and $\lim_{(x,y) \rightarrow (x_0,y_0)} v(x,y) = v_0$ if and only if $\lim_{z \rightarrow z_0} f(z) = w_0$, where $f(z) = u(x,y) + iv(x,y)$, $z = x + iy$ and $z_0 = x_0 + iy_0$, $w_0 = u_0 + iv_0$. 5
- (b) Show that when $T(z) = \frac{az+b}{cz+d}$ ($ad - bc \neq 0$) $2\frac{1}{2} \times 2 = 5$
- (i) $\lim_{x \rightarrow \infty} T(z) = \infty$ if $c = 0$
- (ii) $\lim_{x \rightarrow \infty} T(z) = \frac{a}{c}$ and $\lim_{x \rightarrow d/c} T(z) = \infty$ if $c \neq 0$.
- (c) Show that the function $f(z) = \frac{1}{z^4}$ is differentiable everywhere except at $z \neq 0$. Also, compute $f'(z)$. 4

UNIT-IV

7. (a) If $f'(z) = 0$ everywhere in a domain D , then show that $f(z)$ must be constant throughout D . 5
- (b) Evaluate $\int_C \frac{dz}{z}$, where C represents a square described in the positive sense with sides parallel to the axes and of length $2a$ and having its center at the origin. 5
- (c) Define $\sinh z$ and $\cosh z$ for a complex variable z . Show that $|\cosh z|^2 = \sinh^2 x + \cos^2 y$, whenever $z = x + iy$. $1+3=4$
8. (a) Find the value of the integral $\int_0^{1+i} (x - y + iy) dz$: $2+3=5$
- (i) Along the straight line from $z = 0$ to $z = 1 + i$.
- (ii) Along the real axis from $z = 0$ to $z = 1$ and then along a line parallel to the imaginary axis from $z = 1$ to $z = 1 + i$.

- (b) If $w(t)$ is a piecewise continuous complex-valued function defined on the interval $a \leq t \leq b$ show that

$$\left| \int_a^b w(t) dt \right| \leq \int_a^b |w(t)| dt. \quad 5$$

- (c) Show that $\text{Log}(1+i)^2 = 2\text{Log}(1+i)$, but

$$\text{Log}(-1+i)^2 \neq 2\text{Log}(-1+i). \quad 4$$

UNIT-V

9. (a) If f is analytic everywhere inside and on a simple closed contour C , taken in the positive sense and if z_0 is any point interior to C ,

then show that $f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz.$ 4

- (b) Expand $\frac{1}{z(z^2 - 3z + 2)}$ for 2×3=6

(i) $0 < |z| < 1$ (ii) $1 < |z| < 2$ (iii) $|z| > 2$

- (c) By differentiating the Maclaurin series representation

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \text{ obtain the expressions } \frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} (n+1)z^n$$

$$(|z| < 1) \text{ and } \frac{2}{(1-z)^3} = \sum_{n=0}^{\infty} (n+1)(n+2)z^n \quad (|z| < 1) \quad 4$$

10. (a) Suppose that $z_n = x_n + iy_n$ ($n = 1, 2, 3, \dots$) and $z = x + iy$. Show that $\lim_{n \rightarrow \infty} z_n = z$ if and only if $\lim_{n \rightarrow \infty} x_n = x$ and $\lim_{n \rightarrow \infty} y_n = y$. 4

- (b) Give two Laurent series expansions in powers of z for the

function $f(z) = \frac{1}{z^2(1-z)}$ specifying the regions in which those

expansions are valid. 3+3=6

- (c) Prove that absolute convergence of a series of complex numbers implies convergence of that series. 4