April 2025
B.A./B.Sc.
Fourth Semester
CORE – 10
MATHEMATICS
Course Code: MAC 4.31
(Ring Theory & Linear Algebra - I)

Total Mark: 70 *Pass Mark: 28*
Time: 3 hours

Answer five *questions, taking* one *from each unit.*

UNIT–I

1. (a) Show that the set of integers $\mathbb{Z}$ under ordinary addition and multiplication is a commutative ring with unity. 5
   (b) Define integral domain and field also giving some examples and prove that every field is an integral domain. Justify whether the converse is true or not. 6
   (c) Give an example of ring elements a and b with the properties that $ab = 0$ but $ba \neq 0$. 3

2. (a) Define subring of a ring and prove or disprove that the sum of two subrings of a ring R is a subring of R. 4
   (b) Define characteristic of a ring and prove that the characteristic of an integral domain is either zero or a prime number. 5
   (c) Prove that a finite integral domain is a field. What can you say if the integral domain is infinite? 5

UNIT–II

3. (a) Define ideal of a ring and show that every ideal of a ring R is a subring of R. What can you say about the converse? 4
   (b) Define maximal ideal of a ring and prove that an ideal M of a commutative ring R with unity is a maximal ideal of R if and only if R/M is a field. 7
   (c) For any positive integer n, show that $n\mathbb{Z}$ is an ideal of $\mathbb{Z}$. 3

4. (a) Give an example to show that the union of two ideals of a ring R may not be an ideal of R and prove that the union of two ideals of a ring R is an ideal of R if and only if one of them is contained in the other. 5

(b) Define prime ideal of a ring and prove that an ideal P of a commutative ring R is a prime ideal of R if and only if R/P is an integral domain. 6

(c) If F is a field, prove that its only ideals are $\{0\}$ and F. 3

UNIT–III

5. (a) Define ring isomorphism and state and prove the first isomorphism theorem on rings. 6

(b) If $R = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mid a,b,c \in \mathbb{Z} \right\}$ prove or disprove that the mapping $f : R \to \mathbb{Z}$ defined by $f\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \right) = a$ is a ring homomorphism. 4

(c) If R and R' are two rings and $f : R \to R'$ is a homomorphism, define kernel of f and prove that it is an ideal of R. 4

6. (a) State and prove the second isomorphism theorem on rings. 6

(b) Define embedding and prove that an integral domain can be embedded in a field. 8

UNIT–IV

7. (a) If V is a vector space having a finite basis, prove that every basis for V contains the same number of vectors. 5

(b) If U and W are subspaces of a finite dimensional vector space V, prove that $\dim(U+W) = \dim U + \dim W - \dim(U \cap W)$. 6

(c) Let V be the vector space of functions from $\mathbb{R}$ to $\mathbb{R}$. Show that $f, g, h \in V$ are linearly independent, where $f(x) = e^{2x}$, $g(x) = x^2$, $h(x) = x$. 3

8. (a) Define linear span and prove that the linear span of any non-empty subset of a vector space is a subspace of the vector space. 5

(b) Prove that the union of two subspaces of a vector space is also a subspace of the vector space if and only if one is contained in the other. 5

(c) If the vectors a, b, c are three linearly independent vectors of a vector space V, prove that $a+b$, $b+c$, $c+a$ are also linearly independent. 4

UNIT–V

9. (a) If $T:\mathbb{R}^3 \to \mathbb{R}^2$ is a map given by $T(x,y,z)=(x,y)\forall(x,y,z)\in\mathbb{R}^3$, show that T is a linear transformation and find the null space of T. 4

(b) Define rank and nullity of a linear transformation and state and prove the rank-nullity theorem. 7

(c) If T and S are invertible linear operators on a vector space V, prove that $(TS)^{-1}=S^{-1}T^{-1}$. 3

10. (a) State and prove the first isomorphism theorem of linear algebra. 6

(b) If the matrix representation of a linear transformation T on the vector space $\mathbb{R}^3(\mathbb{R})$ with respect to the basis $\{(1,0,0),(0,1,0)(0,0,1)\}$ is $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}$, what is the matrix representation of T with respect to the basis $\{(0,1,-1),(1,-1,1)(-1,1,0)\}$? 5

(c) If $T:\mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation given by $T(x,y)=(x+y,x-y,y)$, find the range and rank of T. 3