April 2025
B.A./B.Sc.
Fourth Semester
CORE – 9
MATHEMATICS
Course Code: MAC 4.21
(Riemann Integration & Series of Functions)

Total Mark: 70 *Pass Mark: 28*
Time: 3 hours

Answer five *questions, taking* one *from each unit.*

UNIT–I

1. (a) Define the upper Darboux sum $U(f,P)$ and lower Darboux sum $L(f,P)$ for a bounded function *f* on $[a,b]$. State and prove the criterion for *f* to be Riemann integrable in terms of the upper sum and the lower sum. 1+6=7

(b) Let $f(x)=\begin{cases}1, & x\in\mathbb{Q}\\ 0, & x\neq\mathbb{Q}\end{cases}$ on $[0,1]$. For any partition *P* of [0, 1], show that $L(f,P)=0$ and $U(f,P)=1$. Hence, determine whether *f* is Riemann integrable on $[0,1]$. 7

2. (a) Let $f:[a,\ b]\to R$ be a bounded function. Suppose there exist sequences $\{U_n\}$ and $\{L_n\}$ of upper and lower Darboux sums, respectively, such that:

$$\lim_{n\to\infty}(U_n-L_n)=0.$$

Then prove that *f* is Riemann integrable on $[a,b]$, and

$$\int_a^b f=\lim_{n\to\infty}U_n=\lim_{n\to\infty}L_n .$$ 7

(b) Using the definition of the Riemann integral, evaluate $\int_0^1 x^2dx$ by expressing it as the limit of a Riemann sum. 7

UNIT–II

3. (a) Let $f, g : [a, b] \to \mathbb{R}$ be bounded and Riemann integrable functions.

   (i) If $f(x) \le g(x)$ for all $x \in [a, b]$, then prove that

   $$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$ 3½

   (ii) If g is continuous and nonnegative on $[a,b]$, and if

   $$\int_a^b g(x)\,dx = 0,$$

   then prove that $g(x) = 0$ for all $x \in [a,b]$. 3½

   (b) Let f be a continuous function on $\mathbb{R}$, and define the function: $F(x) = \int_{x-1}^{x+1} f(t)\,dt$ for all $x \in \mathbb{R}$. Show that F is differentiable on $\mathbb{R}$, and compute $F'(x)$. 7

4. (a) State and prove the first fundamental theorem of calculus. 2+5=7

   (b) Let f be a bounded function on $[a,b]$, so that there exists $B > 0$ such that

   $$|f(x)| \le B \text{ for all } x \in [a,b].$$

   (i) Show that for all partitions P of $[a,b]$,

   $$U(f^2, P) - L(f^2, P) \le 2B\,\left(U(f,P) - L(f,P)\right).$$ 3

   (ii) Show that if f is integrable on $[a,b]$, then f^2 is also integrable on $[a,b]$. 4

UNIT–III

5. (a) Examine the convergence of the improper integral $\int_0^1 \frac{1-\cos x}{x^p}\,dx$ for various values of $p > 0$. 7

(b) Determine whether the following integrals converge or diverge. 3½×2=7

(i) $\int_1^{\infty} \frac{\ln x}{x} dx$

(ii) $\int_2^{\infty} \frac{dx}{x(\ln x)^2}$

6. (a) Define the gamma function $\Gamma(t)$ and prove the identity $\Gamma(t+1) = t\,\Gamma(t)$. Hence, or otherwise, evaluate $\Gamma(1/2)$. 4+3=7

(b) Show that $\int_0^{\frac{\pi}{2}} (\sin x)^{2m-1} (\cos x)^{2n-1} dx = \frac{1}{2} B(m,n)$ for positive integers *m*, *n*. Compute the integral

$$\int_0^{\frac{\pi}{2}} (\sin x)^5 (\cos x)^3 dx .$$

4+3=7

UNIT–IV

7. (a) Prove that a sequence of functions $\{f_n(x)\}$ defined on a set *S* converges uniformly on *S* if and only if it is uniformly Cauchy. 7

(b) Consider the series

$$\sum_{n=1}^{\infty} x^n (1-x),\ x \in [0,1].$$

Show it converges pointwise on $[0,1]$ and check if it is uniformly convergent there. 7

8. (a) Discuss the pointwise and uniform convergence of the sequence (g_n) where $g_n(x) := \frac{x}{1+nx^2}$ and $g(x) = 0$ for $x \in \mathbb{R}$. 7

(b) Show that $\sum_{n=1}^{\infty} \frac{1}{1+x^n}$ is continuous for $x > 1$. 7

UNIT–V

9. (a) Find the radius and interval of convergence of each of the following: 3½×2=7

(i) $\sum_{n=1}^{\infty} n!x^n$

(ii) $\sum_{n=1}^{\infty} \frac{n+1}{3^n} x^n$

(b) State Abel's theorem. Use it to analyse $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$ on $[-1, 1]$. 2+5=7

10. (a) Let the sequence $\{a_n\}$ be defined by

$$a_n = (-1)^n \cdot \frac{n}{n + 1}.$$

(i) Find $\lim_{x\to\infty} \sup a_n$ and $\lim_{x\to\infty} \inf a_n$ 3

(ii) Determine whether the sequence $\{a_n\}$ converges. Justify your answer. 2

(iii) Find the set of all subsequential limits of the sequence $\{a_n\}$ 2

(b) State the Cauchy-Hadamard theorem. Apply it to $\sum_{n=1}^{\infty} \frac{n!}{n^n} x^n$. 2+5=7