2022

M.Sc.

Fourth Semester

DISCIPLINE SPECIFIC ELECTIVE – 03

MATHEMATICS

Course Code: MMAD 4.11

(Differential Geometry of Manifolds)

Total Mark: 70 *Pass Mark: 28*

Time: 3 hours

Answer five *questions, taking* one *from each unit.*

UNIT–I

1. (a) Define tangent space and vector field and prove that the tangent space $T_p(M)$ on an n-dimensional C^∞ manifold M_n at x is an n-dimensional vector space over $\mathbb{R}$. 2+2+4=8

(b) Define Jacobian map and prove that if $\Phi : M_n \to M_m$ is C^∞ and X, Y are C^∞ vector fields on M_n then $[X, Y]$ is Φ-related. 3+3=6

2. (a) Let Φ and Ψ are given by 3×2=6

(i) $\Phi = xdx - ydy$, $\Psi = zdx + xdz$ and let $\theta = zdy$, then compute $\theta \wedge \Phi \wedge \Psi$

(ii) $\Phi = 6dx \wedge dy + 27dx \wedge dz$, $\Psi = dx + dy + dz$ then compute $\Phi \wedge \Psi \cdot$

(b) Define exterior derivative. If A is an r-form such that $A = A_{i_1 \ldots i_r} dx^{i_i} \wedge \ldots \wedge dx^{i_r}$, then prove that $dA = \partial_j A_{i_1 \ldots i_r} dx^j \wedge dx^{i_i} \wedge \ldots \wedge dx^{i_r}$ and if $A \in V_{\wedge r}$ (set of all C^∞ r-form), then show that dA is skew symmetric. 8

UNIT–II

3. (a) Define Lie group and Lie algebra in detail and prove that the vector space $\mathbb{R}^3$ with the operation cross product of vectors is a Lie algebra. 3+3+2=8

   (b) Define one-parameter subgroups and exponential maps. Let A and B be an infinitesimal transformations of $R_{g(t)}$ and $L_{g(t)}$ respectively. Then show that A is left invariant and B is right invariant, and $A_e = B_e = g(0)$ holds where $g(0)$ is the tangent vector to the curve $g = g(t)$ at $g(0)$. 6

4. (a) Suppose $f : G \to H$ is a Lie group homomorphism. Then prove that the induced map $f_* : T_e(G) \to T_{e'}(H)$ is a homomorphism between the Lie algebras of the groups. If *f* is a Lie group isomorphism, then f_* is an isomorphisms between the Lie algebras, where $e' = f(e) \in H$ for $e \in G$. 4

   (b) Define induced bundle and associate bundle and show that the tangent bundle $T(M)$ and the tensor bundle T_s^r are the associate bundles of $L(M)$. 3+3+4=10

UNIT–III

5. (a) Define Linear connection and prove that $\Gamma_z^{\ h}$ is isomorphic to the tangent space T_x of M_n at $x = \Pi_L(z)$ by the differential map Π_{L*} of the projection Π_L 7

   (b) Define Torsion and curvature form and then state and prove the Bianchi identities for a linear connection. 7

6. (a) State and prove Ricci identity. 6

(b) If L_X denotes the Lie derivative then prove that

(i) $L_{X+Y} = L_X + L_Y$

(ii) $L_X L_Y - L_Y L_X = L_{[X,Y]}$ 8

UNIT–IV

7. (a) Write in brief the definitions of Riemannian manifold and Riemannian connection. Let (M, g) be a Riemannian manifold of dimension n with the Levi-Civita connection $\bar{D}$. If the manifold admits another metric connection D with nonvanishing torsion T, then prove that for any $X,Y,Z \in \mathcal{F}(M)$, we have

$D_X Y - \bar{D}_X Y = \frac{1}{2}\left[T(X,Y) + T'(X,Y) + T'(Y,X)\right]$, where

$g(T(Z,X),Y) = g(T'(X,Y),Z)$ 8

(b) Define curvature tensor and state and prove only second Bianchi's identity. 6

8. (a) Define Ricci tensor and show that it is symmetric. 4

(b) State and prove Schur's theorem. 5

(c) Show that a necessary and sufficient condition that a Riemannian manifold M_n is of constant curvature is that it is projectively flat. 5

UNIT–V

9. (a) Discuss in detail the submanifolds of a manifold. 6

(b) Define induced connection. If $BD_X Y$ is the tangential component of $\bar{D}_{BX} BY$, then prove that D is a covariant derivative. 4

(c) State and prove Gauss formula related to the submanifold. 4

10. (a) Define lines of curvature and principal directions for a curve on a submanifold, and prove the necessary and sufficient condition that the

covariant derivative of a normal vector to a submanifold along a curve in it be tangent to the curve is that the curve is a line of curvature. 7

(b) Define mean curvature of a submanifold and prove that the mean curvature vector field is $\sum_{v=n+1}^{m} \left(C_1^1 Hv\right) Nv$ which is invariant of Nv and show that $Mv = -divNv$, where M is the mean curvature vector of the submanifold M_n 7
