2022

M.Sc.

Fourth Semester

CORE – 12

MATHEMATICS

Course Code: MMAC 4.21

(Rings & Modules)

Total Mark: 70 *Pass Mark: 28*

Time: 3 hours

Answer five *questions, taking* one *from each unit.*

UNIT–I

1. (a) Prove that every field is a division ring and every division ring is an integral domain but not conversely. 4

(b) If R and S are two rings, prove that R×S is also a ring under coordinatewise addition and multiplication. 6

(c) If R is a commutative ring with 1, prove that $A \in M_n(R)$ is a unit if and only if its determinant det(A) is a unit in R 4

2. (a) Define local ring and prove that the characteristic of a local ring is either 0 or a power of a prime. 4

(b) Define endomorphism ring of an abelian group G, $End_{\mathbb{Z}} G$ and prove that $End_{\mathbb{Z}} G$ is a ring under pointwise addition and composition of maps. 6

(c) Prove that the product of two left ideals of a ring R is also a left ideal of R. 4

UNIT–II

3. (a) Define unitary module and prove that unitary modules over $\mathbb{Z}$ are simply abelian groups. 5

(b) For an abelian group M, let $\mathrm{End}_{\mathbb{Z}}(M)$ be the ring of all additive endomorphisms of M. If R is any ring, then prove that M is a left R-module if and only if there exists a homomorphism of rings

$\psi : R \to \mathrm{End}_{\mathbb{Z}}(M)$ 6

(c) Prove that the sum of two submodules of a module M is also a submodule of M. 3

4. (a) State and prove the Epimorphism theorem for modules. 6

(b) Prove that M is a left R-Module if and only if M is a right R^{op}-module, where R^{op} is the ring opposite to R. 5

(c) If M and N are R-modules and $f : M \to N$ is a module homomorphism, define kernel of f and show that it is a submodule of M. 3

UNIT–III

5. (a) If R is a ring and M, N are left R-modules, prove that the Cartesian product $M \times N$ is also a R-module. 6

(b) If R is an integral domain and M is an R-module, show that the set T consisting of the torsion elements of M is a submodule of M and the quotient module $\frac{M}{T}$ is torsion-free. 6

(c) Show that the $\mathbb{Z}$-module $\frac{\mathbb{Z}}{n\mathbb{Z}}(n > 1)$ is not projective. 2

6. (a) Prove that every free module is projective. Is the converse true? Justify. 5

(b) Prove that an R-module P is projective if and only if every exact sequence of the form $0 \to M' \to M \xrightarrow{p} P \to 0$ splits. 6

(c) Prove that an R-module P is projective if and only if P is isomorphic to a direct summand of a free R-module. 3

UNIT–IV

7. (a) If $\{E_i\}_{i\in I}$ is a family of R-modules, prove that $\prod_{i\in I} E_i$ is injective if and only if each E_i is injective. 6

   (b) Prove that every injective module is divisible. 4

   (c) State and prove Schur's lemma. 4

8. (a) Prove that an R-module Q is injective if and only if every exact sequence of the form $0 \to Q \xrightarrow{u} M \to M'' \to 0$ splits. 6

   (b) If the $\mathbb{Z}$-module Q is injective, prove that the left R-module $\mathrm{Hom}_{\mathbb{Z}}(R, Q)$ is injective. 4

   (c) Prove that every quotient of a semi-simple module is semi-simple. 4

UNIT–V

9. (a) Define Artinian module. If a module M is such that it has a submodule N with both N and $\frac{M}{N}$ Artinian, prove that M is Artinian. 5

   (b) Prove that a module is of finite length if and only if it is both Artinian and Noetherian. 5

   (c) Prove that a quotient ring of an Artinian ring is Artinian whereas a subring of an Artinian ring need not be Artinian. 4

10. (a) Prove that sums and direct sums of finitely many Noetherian modules are Noetherian. 5

    (b) Define Nil Radical and Jacobson Radical. Prove that $J_\ell(R) = \{x \in R : 1 - yx \text{ is a unit } \forall y \in R\}$, where $J_\ell(R)$ denotes the left Jacobson Radical. 5

    (c) Prove that matrix rings over Noetherian rings with unity are Noetherian. 4