2022

M.Sc.

Second Semester

CORE – 08

MATHEMATICS

Course Code: MMAC 2.41

(Complex Analysis)

Total Mark: 70 *Pass Mark: 28*

Time: 3 hours

Answer five *questions, taking* one *from each unit.*

UNIT–I

1. (a) Suppose that $z_n = x_n + iy_n \left(n = 1, 2, ...\right)$ and $S = X + iY$, then prove that $\sum_{n=1}^{\infty} z_n = S$ if and only if $\sum_{n=1}^{\infty} x_n = X$ and $\sum_{n=1}^{\infty} y_n = Y$. 5

(b) If a series $\sum_{n=0}^{\infty} a_n \left(z - z_0\right)^n$ converges to $f\left(z\right)$ at all points interior to some circle $\left|z - z_0\right| = R$, then prove that it is the Taylor series expansion for $f\left(z\right)$ in powers of $\left(z - z_0\right)$. 5

(c) If $\sum_{n=0}^{\infty} a_n z^n$ is a power series and let $\sum_{n=1}^{\infty} na_n z^{n-1}$ be the power series obtained by differentiating the first series term by term. Prove that these two series have the same radius of convergence. 4

2. (a) Find the radii of convergence of the following series. 2½×2=5

(i) $\frac{1}{2} z + \frac{1.3}{2.5} z^2 + \frac{1.3.5}{2.5.8} z^3 + ...$

(ii) $\sum_{n=1}^{\infty} \frac{n!}{n^n} z^n$

(b) If a power series $\sum_{n=0}^{\infty} a_n \left(z - z_0\right)^n$ converges when $z = z_1 \left(z_1 \neq z_0\right)$, then show that it is absolutely convergent at each point z in the open disc $|z-z_0| < R_1$, where $R_1 = |z-z_0|$. 5

(c) From the Maclaurin series representation $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \; (|z|<1)$ derive $\frac{1}{z} = \sum_{n=0}^{\infty} (-1)^n (z-1)^n \; (|z-1|<1)$. Hence, find a series representation for $\frac{1}{z^2}$ 4

UNIT–II

3. (a) Represent the function $f(z) = \frac{4z+3}{z(z-3)(z+2)}$ in Laurent's series

 (i) within $|z|=1$ 1½

 (ii) in the annular region between $|z|=2$ and $|z|=3$ 1

 (iii) exterior to $|z|=3$ 1½

 (b) Prove that a function f that is analytic at a point z_0 has a zero of order m there if and only if there is a function g which is analytic and nonzero at z_0, such that $f(z) = (z-z_o)^m g(z)$. 5

 (c) Determine the residues of the multivalued function $f(z) = \frac{z^{\frac{1}{2}}}{z^2+1}$ at each of its poles. 5

4. (a) Explain the various types of singularities with an example each. 4

 (b) If a function $f(z) = \frac{p(z)}{q(z)}$, $p(z)$ and $q(z)$ are analytic at z_0, $p(z_0) \neq 0$, then prove that at the point $z = z_0$, $f(z)$ has a pole of order m if and only if $q(z)$ has a zero of order m. 5

 (c) Compute the residues at singular points of the function $\frac{\cot \pi z}{(z-a)^2}$ 5

UNIT–III

5. Prove the following using contour integration: 7×2=14

(i) $\int_0^\infty \frac{\sin x dx}{x\left(x^2+a^2\right)} = \frac{\pi}{2a^2}\left(1-e^{-a}\right), (a>0)$

(ii) $\int_{-\infty}^\infty \frac{dx}{\left(x^2+b^2\right)\left(x^2+c^2\right)^2} = \frac{\pi(b+2c)}{2bc^2(b+c)^2}, (b>0, c>0)$

6. (a) Prove that $\int_0^\pi \frac{1+2\cos\theta}{5+4\cos\theta} d\theta = 0$ 7

(b) Use residue to compute $\int_0^\infty \frac{x^6}{\left(x^4+a^4\right)^2} dx, (a>0)$ 7

UNIT–IV

7. (a) State and prove the Rouche's theorem. 5

(b) Evaluate the integral $\int_C \frac{f'(z)}{f(z)} dz$ when $f(z) = \frac{\left(z^2+1\right)^2}{\left(z^2+3z+2\right)^3}$ and C is the circle $|z|=3$, taken in the positive sense. 5

(c) Using Rouche's theorem, determine the number of zeros of the polynomial $P(z) = z^{10} - 6z^7 + 3z^3 + 1$ in $|z|=1$ 4

8. (a) State and prove the argument principle. 8

(b) Prove that the equation $e^{z+2} - 2z^7 = 0$ has no solution inside $|z|=1$ 3

(c) Using argument principle, compute $\int_{|z|=3} \frac{z^2}{z^3-2} dz$ 3

UNIT–V

9. (a) Show that the map $w = \frac{1}{z}$ transforms circles and lines into circles and lines. 5

(b) Prove that the set of all bilinear transformations forms a non-abelian group under the composition of transformation. 5

(c) Prove that a mapping $w = f(z)$ is conformal at a point z_0 if it is analytic at z_0 and $f'(z_0) \neq 0$ 4

10. (a) Find the image of the infinite strip $\frac{1}{4} < y < \frac{1}{2}$ under the transformation $w = \frac{1}{z}$. Show the region graphically. 5

(b) Prove that the cross ratio is invariant under bilinear transformation. 5

(c) Find the bilinear transformation which maps $i, 1, -1$ onto $1, 0, \infty$. 4
