2022
M.Sc.
Second Semester
MATHEMATICS
CORE – 07
Course Code: MMAC 2.31
(Number Theory)

Total Mark: 70 *Pass Mark: 28*
Time: 3 hours

Answer five questions, taking one from each unit.

UNIT – I

1. (a) Prove/disprove: The only prime number of the form $n^3 - 1$ is 7. 4
   (b) Is $111^{333} + 333^{111}$ divisible by 7? Justify. 5
   (c) Obtain three consecutive integers, each having a square factor. 5

2. (a) What is the remainder when 18! is divided by 437? 5
   (b) Find the smallest integer $a > 7$ such that
   $2 \mid a,\ 3 \mid a+1,\ 4 \mid a+2,\ 5 \mid a+3,\ 6 \mid a+4$ 6
   (c) If $(a, 35) = 1$, show that $a^{12} \equiv 1 \pmod{35}$. 3

UNIT – II

3. (a) Prove that if a has order hk modulo n, then a^h has order k modulo n. 5
   (b) Show that if p is an odd prime greater than 3, then
   $$\left(\frac{3}{p}\right) = \begin{cases} 1 & \text{if } p \equiv \pm 1 \pmod{12} \\ -1 & \text{if } p \equiv \pm 5 \pmod{12} \end{cases}$$ 9

4. (a) Prove that if n has a primitive root, then it has exactly $\phi(\phi(n))$ of them. 5
   (b) State and prove Gauss's lemma. 9

UNIT – III

5. (a) Prove that in a primitive Pythagorean triple x, y, z, the product xy is divisible by 12 and xyz is divisible by 60. 5

(b) Prove that $x^4 + y^4 = z^2$ has no solution in positive integers. 9

6. (a) In a primitive Pythagorean triple x, y, z, show that $x \neq y \pmod 2$. 4

(b) Show that 3, 4, 5 is the only primitive Pythagorean triple involving consecutive positive integers. 4

(c) Prove the area of a Pythagorean triangle can never be equal to a perfect integral square. 6

UNIT – IV

7. (a) Show that the sum of the squares of the first n Fibonacci numbers is given by $u_1^2 + u_2^2 + \cdots + u_n^2 = u_n u_{n+1}$ 7

(b) Using induction on the positive integer n, show that

$$u_1 + 2u_2 + 3u_3 + \cdots + nu_n = (n+1)u_{n+2} - u_{n+4} + 2$$ 7

8. (a) For the Fibonacci sequence, show that $u_3, u_6, u_9, \ldots$ are all even integers. 7

(b) Show that $u_1 + u_3 + u_5 + \cdots + u_{2n-1} = u_{2n}$ 7

UNIT – V

9. (a) Express $\frac{71}{55}$ as a finite simple continued fraction. 3

(b) If $r = [a_0; a_1, a_2, \ldots, a_n]$, where $r > 1$, show that

$$\frac{1}{r} = [0; a_0, a_1, \ldots, a_n]$$ 5

(c) Find the simple conitinued fraction representations of 3.1416 and 3.14159. 6

10. (a) Compute the convergents of the simple continued fraction $\left[-3;1,1,1,1,3\right]$ 5

(b) Evaluate $p_k, q_k, C_k \left(k-0,1,...8\right)$ for the simple continued fraction $\left[1;2,2,2,2,2,2,2,2\right]$ 9
