2022
M.Sc.
Second Semester
MATHEMATICS
CORE – 06
Course Code: MMAC 2.21
(Measure Theory)

Total Mark: 70 *Pass Mark: 28*
Time: 3 hours

Answer five *questions, taking* one *from each unit.*

UNIT – I

1. (a) Let $\mathcal{A}$ be an algebra of subsets of a set X. Prove that if $A, B \in \mathcal{A}$ then $A \cap B \in \mathcal{A}$ and $A \Delta B \in \mathcal{A}$. 4

   (b) In a σ-algebra $\mathcal{A}$, prove that $\liminf_{n\to\infty} A_n$, $\limsup_{n\to\infty} A_n$ and $\lim_{n\to\infty} A_n$ (if it exists) are in $\mathcal{A}$, where $(A_n : n \in \mathbb{N})$ is any sequence of sets from $\mathcal{A}$. 5

   (c) If $(A_n : n \in \mathbb{N})$ is a monotonic sequence in σ-algebra $\mathcal{A}$, calculate $\liminf_{n\to\infty} A_n$, $\limsup_{n\to\infty} A_n$. Is it true $\lim_{n\to\infty} A_n$ exists in this case? 5

2. (a) Let f be a mapping of a set X into a set Y. If $\mathcal{B}$ is a σ-algebra of subsets of Y then prove that $f^{-1}(\mathcal{B})$ is a σ-algebra of subsets of X. 6

   (b) Let f be a mapping of a set X onto Y. For an arbitrary collection $\mathcal{E}$ of subsets of Y, prove that $\sigma\left(f^{-1}(\mathcal{E})\right) = f^{-1}\left(\sigma(\mathcal{E})\right)$ 8

UNIT – II

3. (a) Define measure μ on a σ-algebra of subsets of X and prove that μ is finitely additive. 2+4=6

(b) Let μ be a measure on a σ-algebra $\mathcal{A}$ of subsets of a set X and $(E_n : n \in \mathbb{N})$ be a decreasing sequence in $\mathcal{A}$ such that there exists a set $A \in \mathcal{A}$ with $\mu(A) < \infty$ and $E_1 \subset A$. Prove that

$$\lim_{n\to\infty} \mu(E_n) = \mu\left(\lim_{n\to\infty} E_n\right)$$ 8

4. (a) Define the outer measure μ^* on $\mathcal{P}(X)$ and prove that μ^* is additive on $\mathcal{M}(\mu^*)$. 2+3+2=7

(b) Let μ^* be regular and σ-finite outer measure on a set X. Prove that the following two conditions are equivalent.

(i) $\mathcal{M}(\mu^*) = \mathcal{P}(X)$

(ii) $E \in \mathcal{P}(X), F \in \mathcal{M}(\mu^*), E \subset F, \mu^*(E) = \mu^*(F)$ implies $\mu^*(F - E) = 0$. 7

UNIT – III

5. (a) Define Lebesgue outer measure μ_L^* on $\mathbb{R}$ and prove that $\mu_o^*(E) = \mu_c^*(E)$ for every $E \in \mathcal{P}(\mathbb{R})$. 2+5=7

(μ_o^*, μ_c^* denote Lebesgue outer measure using open and closed intervals respectively.)

(b) Prove that every interval in $\mathbb{R}$ is Lebesgue outer measurable. 7

6. (a) For the Lebesgue measure space $(\mathbb{R}, \mathcal{M}_L, \mu_L)$ prove the following.

(i) $(\mathbb{R}, \mathcal{M}_L, \mu_L)$ is σ-finite 2

(ii) Every Borel set in $\mathbb{R}$ is a Lebesgue measurable set 3

(iii) Every non-empty open set O in $\mathbb{R}$, $\mu_L(O) > 0$ 2

(b) For $E \in \mathcal{P}(\mathbb{R})$, prove that the following conditions are equivalent.

(i) $E \in \mathcal{M}_L$

(ii) For every $\varepsilon > 0$, there exists a closed set $C \subset E$ with $\mu_L^*(E - C) \le \varepsilon$ 7

(You may assume the result that you may be using in proving the above result.)

UNIT – IV

7. (a) Let $(X, \mathcal{A})$ be a measurable space and let f be an extended real valued $\mathcal{A}$-measurable function defined on $D \in \mathcal{A}$. Prove that

   (i) $\{x \in D / f(x) = \alpha\} \in \mathcal{A}$ for every $\alpha \in \overline{\mathbb{R}}$

   (ii) $\{x \in D / f(x) \in \mathbb{R}\} \in \mathcal{A}$ 5+2=7

   (b) Let $(X, \mathcal{A})$ be a measurable space and let f and g be two extended real valued $\mathcal{A}$-measurable functions on a set $D \in \mathcal{A}$. Find the domain of definition $\mathcal{D}(fg)$ of fg and prove that $\mathcal{D}(fg)$ is in $\mathcal{A}$ and fg is $\mathcal{A}$-measurable on the domain. 7

8. (a) Define convergence *a.e.* (almost everywhere) and prove that if $(f_n : n \in \mathbb{N})$ be a sequence of real valued $\mathcal{A}$-measurable functions on a set $D \in \mathcal{A}$ and g_1, g_2 are $\mathcal{A}$-measurable on D and $\lim_{n\to\infty} f_n = g_1$ *a.e.* and $\lim_{n\to\infty} f_n = g_2$ *a.e.*, then $g_1 = g_2$ *a.e.* on D 7

   (b) State and prove Egorov theorem. 7

UNIT – V

9. (a) Define a simple function, express its canonical representation and its Lebesgue integral. 2+2+3=7

   (b) State and prove Fatou's Lemma for non-negative measurable functions. 7

10. (a) If f is integrable on D, then prove that then $|f| < \infty$ on D. 7

    (b) Let $(X, \mathcal{A}, \mu)$ be a measure space. Let f and g be two extended real valued $\mathcal{A}$-measurable functions on $D \in \mathcal{A}$. Suppose $f \le g$. Prove that

    (i) if f is semi integrable on D and $\int_D f d\mu \neq -\infty$, then g is semi-integrable on D.

(ii) if g is semi-integrable on D and $\int_D g d\mu \neq \infty$, then f is semi-integrable on D. 7
