2022

B.A./B.Sc.

Sixth Semester

DISCIPLINE SPECIFIC ELECTIVE – 4

MATHEMATICS

Course Code: MAD 6.21

(Differential Geometry)

Total Mark: 70 *Pass Mark: 28*

Time: 3 hours

Answer five questions, taking one from each unit.

Note: Symbols have their usual meaning.

UNIT–I

1. (a) Define osculating plane and find the equation of the osculating plane at the point of the curve. 4

   (b) Define curvature and find an expression for the curvature at a given point to a given curve. Also show that a necessary and sufficient condition for the curve to be straight line is that the curvature $\kappa = 0$ at all points of the curve. 6

   (c) Find the radius of curvature and radius of torsion at a point of intersection of the surfaces $x^2 + y^2 = z^2, z = a\tan^{-1}\left(\frac{y}{x}\right)$, where $y = x\tan\theta$ 4

2. (a) State and prove the existence theorem for space curves. 7

   (b) Define Osculating circle and osculating sphere and find the equation of the osculating circle at $(1,2,3)$ on the curve $x = 2t+1,\ y = 3t^2+2,\ z = 4t^3+3$ 2+5=7

UNIT – II

3. (a) Let $\overline{r} = \overline{r}(u,v)$ be the equation of the surface. Then prove that a proper parametric transformation either leaves every normal unchanged or reverses every normal. 4

(b) Define second fundamental form and find second order magnitudes of the surface $\overline{r} = \overline{r}(u, v)$ and calculate the fundamental magnitudes and the normal to the surface $2z = ax^2 + 2hxy + by^2$, taking x, y as parameters. 5

(c) Find the angle between two directions on the surface at the point P having direction coefficients (l, m) and (l', m') and find the direction coefficients of the direction making an angle $\frac{\pi}{2}$ with the direction having direction coefficients (l, m). 5

4. (a) Define principal curvature and find the equation giving the principal curvatures. 6

(b) Define lines of curvature on a surface and then state and prove the Rodrigue's formula. 1+7=8

UNIT – III

5. (a) Show that a necessary and sufficient condition for a curve v = constant to be geodesic on the general surface is $EE_2 + FE_1 - 2EF_1 = 0$ 5

(b) Discuss the nature of a geodesics on a surface of revolution. 9

6. (a) State and prove Clairaut's theorem. 6

(b) Find the geodesic curvature of the parametric curve $v = c$. 4

(c) Prove that the curves of the family $\left(v^3 / u^2\right) =$ const. are geodesics on a surface with metric $v^2du^2 - 2uv\,du\,dv + 2u^2dv^2, \left(u, v > 0\right)$ 4

UNIT – IV

7. (a) Define contravariant and covariant vectors and prove that the transformations of these vectors are transitive. 6

(b) Prove that the open product of two vectors is a tensor of order two, and the outer product of two tensors is a tensor whose order is sum of the orders of the two tensors. 2+3=5

(c) Show that Kronecker delta is a mixed tensor of rank two. 3

8. (a) Define inner product of two tensors, then state and prove quotient law of tensors. 6

(b) Let a quantity $A(p,q,r)$ is such that in the x^i coordinate system, $A(p,q,r)B_r^{qs} = C_p^s$, where B_r^{qs} is an arbitrary tensor and C_p^s is a tensor. Prove that $A(p,q,r)$ is a tensor. 4

(c) If $A_{ij} = 0$ for $i \neq j$ and $A_{ij} \neq 0$ for $i = j$, then show that the conjugate tensor $B^{ij} = 0$ for $i \neq j$ and $B^{ii} = \frac{1}{A_{ii}}$ (no summation) 4

UNIT – V

9. (a) Show that the Riemannian metric g_{ij} is a second rank covariant symmetric tensor. 4

(b) Define Christoffel symbols of first and second kind and show that they are not tensor quantities. 6

(c) If ∇^2 is the Laplacian operator and ϕ is a scalar function of coordinates x^i, then prove that

$$\nabla^2\phi = g^{ij}\left(\frac{\partial^2\phi}{\partial x^i \partial x^j} - \frac{\partial\phi}{\partial x^l}\left\{ {l \atop i\,j} \right\}\right) = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\sqrt{g}\;\; g^{ij}\phi_{,j}\right)$$ 4

10. (a) Obtain covariant derivative of a covariant vector with respect to the fundamental tensor g_{ij}. 4

(b) Prove a necessary and sufficient condition that the curl of a vector field vanishes is that the vector field be gradient. 3

(c) If A_{ij} is the curl of a covariant vector, show that

$A_{ij,k} + A_{jk,i} + A_{ki,j} = 0$ and that is equivalent to

$$\frac{\partial A_{ij}}{\partial x^k} + \frac{\partial A_{jk}}{\partial x^i} + \frac{\partial A_{ki}}{\partial x^j} = 0$$ 7