2022

B.A/B.Sc.

Sixth Semester

CORE – 14

MATHEMATICS

Course Code: MAC 6.21

(Ring Theory & Linear Algebra – II)

Total Mark: 70 *Pass Mark: 28*

Time: 3 hours

Answer five *questions, taking* one *from each unit.*

UNIT–I

1. (a) Prove that R is an integral domain if and only if $R[x]$ is an integral domain. If R is a field, will $R[x]$ be a field? Justify. 5

   (b) State and prove the division algorithm for polynomials. 5

   (c) If $\mathbb{R}$ is the field of real numbers, show that $\frac{\mathbb{R}[x]}{\langle x^2+1\rangle} \cong \mathbb{C}$, the ring of complex numbers. 4

2. (a) Let F be a field. If $f(x) \in F[x]$ and $\deg f(x)$ is 2 or 3, prove that $f(x)$ is reducible over F if and only if $f(x)$ has zero in F. 4

   (b) If $f(x) \in \mathbb{Z}[x]$ is reducible over $\mathbb{Q}$ (the field of rationals), prove that $f(x)$ is reducible over $\mathbb{Z}$ (the ring of integers) 5

   (c) If F is a field, prove that $F[x]$ is a principal ideal domain. 5

UNIT – II

3. (a) Prove that every Euclidean domain is a principal ideal domain. 5

(b) Show that $\frac{\mathbb{Q}[x]}{I}$, where $I = \langle x^2 - 5x + 6 \rangle$, is not a field. 2

(c) If R is a unique factorization domain, prove that the polynomial ring $R[x]$ is also a unique factorization domain. 7

4. (a) Prove that any two elements in a Euclidean domain have a greatest common divisor. 5

(b) Prove that in a unique factorization domain, an element is prime if and only if it is irreducible. 5

(c) Find all units of the ring $\mathbb{Z}\left[\sqrt{-5}\right]$. 4

UNIT – III

5. (a) If *W* is a subspace of a finite dimensional vector space *V* over a field *F*, prove that $\dim W + \dim W^o = \dim V$, where W^o is the annihilator of *W*. 6

(b) If $B = \{(2,1),(3,1)\}$ is a basis of $\mathbb{R}^2$ over $\mathbb{R}$, find the dual basis of A. 3

(c) If *T* is a linear operator on the vector space *V* over a field *F* and $\lambda \in F$ is an eigenvalue of *T*, then for any polynomial $f(x) \in F[x]$, show that $f(\lambda)$ is an eigenvalue of $f(T)$. 5

6. (a) If *T* is a linear operator on the vector space *V* over a field *F*, prove that the roots of the minimal polynomial and the characteristic polynomial of T are same, except for their multiplicities. 5

(b) If *T* is a linear operator on $\mathbb{R}^3$ which is represented in the standard ordered basis by the matrix $A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$, prove that *T* is diagonalizable by exhibiting a basis of $\mathbb{R}^3$, each of which is a characteristic vector of T. 6

(c) If T is a linear operator on a vector space V, prove that $\ker T$ and $\operatorname{Im}(T)$ are invariant under T, where $\ker T$ and $\operatorname{Im}(T)$ denote the kernel and image of T respectively. 3

UNIT – IV

7. (a) State and prove Cauchy-Schwarz inequality for inner product space. 5

(b) If W is a non-empty subset of a inner product space V over a field F, prove that the orthogonal complement of W, $W^{\perp}$ is a subspace of V. 3

(c) Find an orthonormal basis of the vector space V of all real polynomials of degree not greater than 2, in which the inner product is defined as $\langle f(x), g(x)\rangle = \int_{-1}^{1} f(x)g(x)\,dx$. 6

8. (a) Prove that every finite dimensional inner product space has an orthonormal basis. 6

(b) Prove that an orthogonal set of nonzero vectors of an inner product space is linearly independent. 4

(c) State and prove the triangle inequality for inner product space. 4

UNIT – V

9. (a) If V is a finite dimensional inner product space and T is linear operator on V, prove that there exists a unique function $T^*: V \to V$ such that $\langle T(x), y\rangle = \langle x, T^*(y)\rangle \;\forall x, y \in V$. Also show that T^* is linear. 5

(b) Prove that every linear operator on a finite dimensional inner product space V can be uniquely expressed as $T = T_1 + T_2$, where T_1 is self-adjoint and T_2 is skew-symmetric. 4

(c) If T is a linear operator on a complex inner product space V, prove that T is normal if and only if $\|T^*(u)\| = \|T(u)\| \; \forall u \in V$, where T^* is the adjoint of T. 5

10. (a) If T is a self-adjoint operator on an inner product space V, prove that every eigenvalue of T is real. Also, prove that eigenvectors u, v of T corresponding to distinct eigenvalues are orthogonal. 6

(b) Let T be a linear operator on an inner product space V and W be a subspace of V. If W is invariant under T, prove that $W^{\perp}$ is invariant under T^*. 3

(c) If T is a normal operator on an inner product space V over a field F, show that $T - \alpha I$ is normal for every $\alpha \in F$, where I is the identity on V. 5