2022

B.A/B.Sc.

Sixth Semester

CORE – 13

MATHEMATICS

Course Code: MAC 6.11

(Metric Spaces and Complex Analysis)

Total Mark: 70 *Pass Mark: 28*

Time: 3 hours

Answer five questions, taking one from each unit.

UNIT–I

1. (a) Prove that, for any two real numbers x and y,

$$\frac{|x+y|}{1+|x+y|} \leq \frac{|x|}{1+|x|} + \frac{|y|}{1+|y|}$$ 6

(b) Let $X = \mathbb{R}^n$. For $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n)$ in $\mathbb{R}^n$, define $d_p(x, y) = \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{\frac{1}{p}}$ where $p \geq 1$. Verify that d_p is a metric on $\mathbb{R}^n$ 8

2. (a) Let (X, d) be a metric space and A, B be subsets of X. Prove the following results. 6

(i) $A \subseteq B \Rightarrow A^o \subseteq B^o$

(ii) $(A \cap B)^o = A^o \cap B^o$

(iii) $(A \cup B)^o \supseteq A^o \cup B^o$

(b) Let (X, d) be a metric space. Prove the following results.

4+2+2=8

(i) If F is a subset of X, then F is closed if and only if $F = \overline{F}$

(ii) If $A \subseteq B$ then $\overline{A} \subseteq \overline{B}$

(iii) If $A \subseteq F$ and F is closed, then $\overline{A} \subseteq F$

UNIT – II

3. (a) Let (X, d_X) and (Y, d_Y) be metric spaces and $A \subset X$. Prove that $f : A \to Y$ is continuous at $a \in A$ if and only if whenever a sequence $\{x_n\}$ in A converges to a, the sequence $\{f(x_n)\}$ converges to $f(a)$ 7

(b) Define disconnectedness of a metric space and prove that (X, d) is disconnected if and only if there exists two non-empty disjoint open sets A and B of X such that $X = A \cup B$ 1+6=7

4. (a) State and prove Banach fixed point theorem. 7

(b) Prove that the function $f : (0,1) \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$ is not uniformly continuous, where the sets $(0,1), \mathbb{R}$ have the standard metric. 7

UNIT – III

5. (a) Derive the Cauchy-Riemann equation for an analytic function $f(z)$.

6

(b) Check if $f(z)$ is continuous at $z=0$ or not.

$$f(z)=\begin{cases}\dfrac{\operatorname{Re}(z^2)}{|z|}; & z\neq 0\\ 0 & ; z=0\end{cases}$$ 4

(c) Show that if limit of a function $f(z)$ exist at a point z_0, then it must be unique. 4

6. (a) If $f(z)$ and $g(z)$ are two functions such that

$\lim_{z\to z_0} f(z)=l_1, \lim_{z\to z_0} g(z)=l_2$ then $\lim_{z\to z_0}(fg)(z)=l_1l_2$. 5

(b) Find the limit $\lim_{z\to -1}\dfrac{iz+3}{z+1}$ 3

(c) Prove that continuity is a necessary but not a sufficient condition for differentiability. 6

UNIT – IV

7. (a) State and prove the Cauchy-Goursat theorem. 10

(b) Evaluate $I=\int_C x\,dz$, where C is the circle $|z|=R$. 4

8. (a) Evaluate $\int_C \dfrac{4-3z}{z(z-1)(z-2)}dz$, where C is $|z-2|=1.5$ 4

(b) Evaluate $I=\int_C x\,dz$, where C is the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ 4

(c) Prove that $|e^z|=1$ if and only if $z\in\mathbb{R}_i$. 3

(d) Solve $\int_C \dfrac{e^{2z}}{(z+1)^4}dz$, where C is $|z|=2$ 3

UNIT – V

9. (a) State and prove the Liouville's theorem. 4

(b) Obtain the Taylor's and Laurent's series that represents the function

$$f(z)=\frac{z^2-1}{(z+2)(z+3)}$$ in the region

(i) $|z|<2$

(ii) $|z|>3$ 6

(c) Prove that $\tan^{-1} z = z-\frac{z^3}{3}+\frac{z^5}{5}-\cdots$ when $|z|<1$ 4

10. (a) Find the radius of convergence of the series

$$\frac{1}{2}z+\frac{1\cdot 3}{2\cdot 5}z^2+\frac{1\cdot 3\cdot 5}{2\cdot 5\cdot 8}z^3+\cdots$$ 4

(b) State and prove the Weierstrass M-test. 4

(c) Obtain the Taylor's or Laurent's series which represent the function

$$f(z)=\frac{1}{(1+z^2)(z+2)}$$ when

(i) $|z|<1$

(ii) $1<|z|<2$ 6