2022

B.A./B.Sc.

Fourth Semester

CORE – 9

MATHEMATICS

Course Code: MAC 4.21

(Riemann Integration & Series of Functions)

Total Mark: 70 *Pass Mark: 28*

Time: 3 hours

Answer five *questions, taking* one *from each unit.*

UNIT–I

1. (a) Let f be a bounded function on $[a,b]$. If P and Q are partitions of $[a,b]$ and $P \subseteq Q$, then prove that

$$L(f,P) \le L(f,Q) \le U(f,Q) \le U(f,P).$$ 7

(b) Define Upper Darboux integral ($U(f)$) and Lower Darboux integral ($L(f)$) and test whether the function

$$f(x)=\begin{cases}1 \text{ if } x \text{ is rational in } [a,b]\\ 0 \text{ if } x \text{ is irrational in } [a,b]\end{cases}$$ is integrable over [a,b] or not.

1½+1½+4=7

2. (a) Prove that a bounded function f on $[a,b]$ is Riemann integrable if and only if it is Darboux integrable in which case the values of the integrals agree. 7

(b) Prove that a bounded function f on $[a,b]$ is integrable if and only if for each $\varepsilon > 0$, there exists a partition P of $[a,b]$ such that

$$U(f,P) - L(f,P) < \varepsilon.$$ 7

UNIT–II

3. (a) If f and g are Riemann integrable on [a, b] then, prove that $f-g$ is also Riemann integrable on $[a, b]$ and $\int_a^b (f-g) = \int_a^b f - \int_a^b g$. 7

(b) Prove that every continuous function f on $[a,b]$ is integrable. 7

4. (a) Let g be continuous function on $[a,b]$ that is differentiable on (a,b) and if g' (derivative of g) is integrable on $[a,b]$, then prove that

$$\int_a^b g' = g(b) - g(a)$$ 7

(b) Show $\left|\int_{-2\pi}^{2\pi} x^4 \sin^6\left(e^{3x}\right) dx\right| \leq \frac{64\pi^5}{5}$ 7

UNIT–III

5. (a) Prove that $\int_a^b \frac{dx}{(x-a)^n}$ converges for $n < 1$ and diverges for $n \geq 1$

5

(b) Prove that $\int_a^\infty \frac{dx}{(x-a)^n} (a > 0)$ converges for $n > 1$ and diverges for $n \leq 1$. 5

(c) Test the convergence of $\int_0^1 \frac{dx}{x^{\frac{1}{3}}(1-x)^{\frac{2}{3}}}$ 4

6. (a) Prove that $\int_0^1 x^{m-1}(1-x)^{n-1} dx$ converges for $m > 0$ and $n > 0$. 5

(b) Prove that $\int_0^{\frac{\pi}{2}} \sin^{2m-1} x \cos^{2n-1} x dx = \frac{1}{2} B(m,n)$ 5

(c) Evaluate (i) $\int_0^\infty \sqrt{x} e^{-x^3} dx$ (ii) $\int_0^{2\pi} \sin^8 x dx$ 2×2=4

UNIT–IV

7. (a) Prove that the sequence of functions $\{f_n\}$, where $f_n(x) = \dfrac{n+x}{3n+2x}$ for $n \in \mathbb{N}$ and $x \geq 0$ converges uniformly on $[0,M]$ for any $M > 0$ and does not converge uniformly on $[0,\infty)$. 7

(b) Let $\{f_n\}$ be a sequence of functions continuous on the interval I and converging uniformly on I to f. Prove that f is continuous on I. 7

8. (a) State Weierstrass M-test and test the convergence of the series

(i) $\sum_{n=0}^{\infty} \left(\dfrac{3x}{4}\right)^n$ in $[-1,1]$ 2+2½+2½=7

(ii) $\sum_{n=0}^{\infty} \dfrac{1}{1+n^2x^2}$ in $[1,\infty)$

(b) Define, for $0 < x < 1$ and $n \in \mathbb{N}$, $f_n(x) = \dfrac{1}{x^2} + \dfrac{1}{n}$ and $g_n(x) = \dfrac{x}{n}$.

(i) Prove that $\{f_n\}$ and $\{g_n\}$ converge uniformly on $(0,1)$

(ii) Does $\{f_n \cdot g_n\}$ converge uniformly on $(0,1)$? Justify. 7

UNIT–V

9. (a) Prove that if $\sum_{n=0}^{\infty} a_n x^n$ converges for $x = c\,(\neq 0)$, then the series converges absolutely for all x with $|x| < |c|$. 8

(b) Prove that 2×3=6

(i) $\sum_{n=0}^{\infty} e^{n^2} x^n$ diverges for all nonzero values of x.

(ii) $\sum_{n=0}^{\infty} \dfrac{x^n}{n!}$ converges for all values of x in $\mathbb{R}$

(iii) $\sum_{n=1}^{\infty} \frac{n(x-3)^3}{(n+1)3^n}$ converges for all values of $x \in (0,6)$.

10. (a) Find the power series expansion in powers on x and the radius of convergence of 4×2=8

(i) $\frac{x}{x^2+5}$

(ii) $\frac{2}{3x+4}$

(b) Let R be the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$, $0 < R < \infty$. What can be said about the radius of convergence of the following series?

(i) $\sum_{n=0}^{\infty} a_n x^{2n}$ 2×3=6

(ii) $\sum_{n=0}^{\infty} a_n^2 x^n$

(iii) $\sum_{n=0}^{\infty} a_n^2 x^{2n}$