2022

B.A/B.Sc.

Fourth Semester

CORE – 10

MATHEMATICS

Course Code: MAC 4.31

(Ring Theory & Linear Algebra – I)

Total Mark: 70 *Pass Mark: 28*

Time: 3 hours

Answer five *questions, taking* one *from each unit.*

UNIT–I

1. (a) Show that a Boolean ring is commutative. 4

(b) Define subring of a ring R and prove that the intersection of two subrings of a ring R is also a subring of R 5

(c) Prove that every finite integral domain is a field. 5

2. (a) Prove that a commutative ring R is an integral domain if and only if for all $a, b, c \in R\left(a \neq 0\right), ab = ac \Rightarrow b = c$ 4

(b) Define centre of a ring R and prove that it is a subring of R 5

(c) Define characteristic of a ring R and prove that the characteristic of an integral domain is either zero or a prime. 5

UNIT – II

3. (a) Define product of two ideals of a ring R and prove that the product of two ideals of a ring R is also an ideal of R. 5

(b) Define prime ideal of a ring R and prove that P is prime ideal of R if and only if $\frac{R}{P}$ is an integral domain. 6

(c) If A is a left ideal and B is a right ideal of a ring R, show that $A+B$ may not be an ideal of R 3

4. (a) Prove that the intersection of two ideals of a ring R is also an ideal of R. 4

(b) Define maximal ideal of a ring R and prove that M is maximal ideal of R if and only if $\frac{R}{M}$ is a field. 6

(c) If S is a non-empty subset of a ring R, show that the right annihilator of S is a right ideal of R. 4

UNIT – III

5. (a) Define ring homomorphism. If R and R' are two rings and $f : R \to R'$ is a ring homomorphism, prove that kernel of f is an ideal of R. 4

(b) State and prove the fundamental theorem of ring homomorphism. 7

(c) If $f : R \to R'$ is an onto homomorphism, where R is a ring with unity, show that f(1) is the unity of R'. 3

6. (a) If $\mathbb{Z}$ is the ring of integers, show that the only homomorphisms from $\mathbb{Z} \to \mathbb{Z}$ are the identity and zero mappings. 3

(b) Prove that an integral domain can be embedded into a field. 8

(c) If $f : R \to R'$ is a ring homomorphism, prove that $\ker f = \{0\}$ if and only if f is one-one, where ker $\ker f$ is the kernel of f. 3

UNIT – IV

7. (a) Prove that the union of two subspaces of a vector space V is a subspace of V if and only if one is contained in the other. 5

(b) Examine whether the vectors $(2,3,-1)$, $(-1,4,2)$, $(1,18,-4)$ are linearly independent or not in $\mathbb{R}^3$ over $\mathbb{R}$. 3

(c) Prove that the set of non-zero vectors $v_1, v_2, ..., v_n$ of a vector space V over a field F is linearly dependent if and only if some $v_r, 2 \le r \le n$ is a linear combination of the preceding ones. 6

8. (a) Prove that there exists a basis for every finite dimensional vector space. 6

(b) Prove that the intersection of two subspaces of a vector space V is also a subspace of V. 4

(c) Show that the set $\{(1,2,1),(2,1,0),(1,-1,2)\}$ forms a basis of $\mathbb{R}^3$ over $\mathbb{R}$. 4

UNIT – V

9. (a) Define linear transformation, null space, range, rank and nullity of a linear transformation. 3

(b) State and prove the Rank-Nullity theorem. 7

(c) Find the matrix representation of the linear map $T:\mathbb{R}^3 \to \mathbb{R}^3$ given by $T(x,y,z)=(z,y+z,x+y+z)$ relative to the basis $\{(1,0,1),(-1,2,1),(2,1,1)\}$. 4

10. (a) If V and W are two vector spaces and $T:V \to W$ is an onto linear transformation, prove that $\frac{V}{\ker T} \cong W$, where ker T is the kernel of T. 6

(b) Prove that two finite dimensional vector spaces are isomorphic if and only if they have the same dimension. 8