2022

B.A./B.Sc.

Second Semester

CORE – 3

MATHEMATICS

Course Code: MAC 2.11

(Real Analysis)

Total Mark: 70 *Pass Mark: 28*

Time: 3 hours

Answer five *questions, taking* one *from each unit.*

UNIT–I

1. (a) State and prove the triangular inequality. 4

   (b) Show that the union of two disjoint denumberable sets is denumerable. 5

   (c) Show that the infimum of $S = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$ is 0. 5

2. (a) Show that the set of even integers is denumerable. 5

   (b) Find all $x \in \mathbb{R}$ that satisfy the inequality $4 < |x+2| + |x-1| < 5$. 4

   (c) Let S be a non-empty set in $\mathbb{R}$ which is bounded above. Prove that an upper bound u of S is the supremum if and only if for every $\varepsilon > 0$, there exists $s \in S$ such that $u - \varepsilon < s$. 5

UNIT–II

3. (a) Let S be a nonempty subset of $\mathbb{R}$ that is bounded below. Prove that $\inf S = -\sup\{-s : s \in S\}$. 7

   (b) Let $A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$. What are the isolated points of A? Justify your answer. 7

4. (a) Prove that every neighbourhood of a limit point of a set contains infintely many points of the set. 6

(b) Prove that $\mathbb{R}$ is not countable. 8

UNIT–III

5. (a) Let (x_n) and (y_n) be sequences converging to x and y respectively. Prove that $(x_n y_n)$ converges to xy. 5

(b) If $0 < a < 1, b > 1$, discuss the convergence of $\left(\frac{b^n}{n!}\right)$. 4

(c) Let $x_1 = 1$ and $x_{n+1} = \sqrt{2 + x_n}$ for $n \in \mathbb{N}$. Show that (x_n) is convergent and find the limit. 5

6. (a) Using the definition of the limit of a sequence, show that

(i) $\lim\left(\frac{2n}{n+1}\right) = 2$ (ii) $\lim\left(\sqrt{n+1} - \sqrt{n}\right) = 0$ 6

(b) State and prove the monotone convergence theorem. 8

UNIT–IV

7. (a) Show that the sequence $\left(1 - (-1)^n + \frac{1}{n}\right)$ is divergent. 5

(b) Prove that every sequence of real numbers has a monotone subsequence. 6

(c) Show that $\left(\frac{1}{n}\right)$ is a Cauchy sequence. 3

8. (a) Show directly from the definition that $\left(1 + \frac{1}{2!} + \cdots + \frac{1}{n!}\right)$ is a Cauchy sequence. 6

(b) Let (x_n) be a Cauchy sequence such that x_n is an integer for every $n \in \mathbb{N}$. Show that (x_n) is ultimately constant. 6

(c) Give two examples of unbounded sequences that have convergent subsequences. 2

UNIT–V

9. (a) Define an infinite series. Give an example each of a convergent and a divergent series. 4

(b) Show that the *p*-series convergess when $p > 1$. 5

(c) If a series is absolutely convergent in $\mathbb{R}$, then prove that it is convergent. Does the converse of he statement hold? Justify. 5

10. (a) Discuss the convergence or divergence of the following series whose *n*th term is: 6

(i) $n^n e^{-n}$ (ii) $\frac{n!}{n^n}$

(iii) $(n \ln n)^{-1}$

(b) If $\sum x_n$ with $x_n > 0$ is convergent, then is $\sum \sqrt{x_n}$ always convergent? Either prove it or give a counter example. 4

(c) Does the series $\sum_{n=0}^{\infty} \frac{\sqrt{n-1} - \sqrt{n}}{\sqrt{n}}$ converge? 4