2021
M.Sc.
Third Semester
DISCIPLINE SPECIFIC ELECTIVE – 02
MATHEMATICS
Course Code: MMAD 3.21
(Tensor Analysis & Riemannian Geometry)

Total Mark: 70 *Pass Mark: 28*
Time: 3 hours

Answer five *questions, taking* one *from each unit.*

UNIT–I

1. (a) Prove the property that tensor law of transformation possesses the group property. 3
   (b) Define contraction of a mixed tensor. By taking one example show that in the process of contraction, the rank of the tensor is reduced by two. 6
   (c) Prove that the set of n^3 functions A^{ijk} form the components of a tensor if $A^{ijk}B^p_{ij} = C^{pk}$ provided that B^p_{ij} is an arbitrary tensor and C^{pk} is a tensor. What happens if B^p_{ij} is symmetrical in i and j? 5
2. (a) Define inner product of vectors. Prove that the inner product of covariant and contravariant vectors is a scalar invariant. Also state and prove the quotient law. 6
   (b) If A^i is an arbitrary contravariant vector and $C_{ij}A^iA^j$ is an invariant, then show that $C_{ij}+C_{ji}$ is a covariant tensor of second order. 5
   (c) Prove that $A_{ij}B^iC^j$ is invariant if B^i and C^j are vectors and A^{ij} is a tensor. 3

UNIT–II

3. (a) Show that g_{ij} is a second rank covariant symmetric tensor. 6
   (b) Show that the angle between the contravariant vectors is real when the Riemannian metric is postive definite. 4
   (c) Determine the metric tensor and its conjugate (reciprocal) tensor in cylindrical co-ordinates. 4

4. (a) Prove that the necessary and sufficient condition for the existence of an n-ply orthogonal system of co-ordinates hypersurfaces is that the fundamental form must be of the form $ds^2 = \sum_{i=1}^{n} g_{ii}(dx^i)^2$. 5

(b) Prove that the inclination θ of the two vectors has the same value whether they are regarded as vectors in V_n or as vectors in a Euclidean space S_m in which V_n is immersed. 5

(c) Find out the line element on the surface of a sphere in a V_2. 4

UNIT–III

5. (a) Define Christoffel symbols and obtain tensor laws of transformations of these symbols. 6

(b) Prove that the laws of transformations of Christoffel symbols possess the group property. 5

(c) Prove that if A^{ij} is a symmetric tensor, then

$$A^j_{i,j} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^j}\left(A^j_i\sqrt{g}\right) - \frac{1}{2}A^{jk}\frac{\partial g_{jk}}{\partial x^i}.$$ 3

6. (a) Obtain the covariant derivative of A^{ij}. 6

(b) If ϕ is a scalar function of x^i, then prove that

$$\nabla^2\phi = g^{ij}\left(\frac{\partial^2\phi}{\partial x^i \partial x^j} - \frac{\partial\phi}{\partial x^l}\begin{Bmatrix} l \\ i\ j \end{Bmatrix}\right) = \frac{1}{\sqrt{g}}\partial_i\left(\sqrt{g}\, g^{ij}\partial_j\phi\right).$$ 5

(c) Prove that $u.\nabla u = -u.curl\, u$ if u is a vector of constant magnitude. 3

UNIT–IV

7. (a) Define geodesic. Find the differential equations of a geodesic using the property that it is a path of maximum or minimum length joining the two points on it. 8

(b) Prove the necessary and sufficient conditions that the hypersurfaces ϕ = constant form a system of parallels is that $(\nabla\phi)^2 = 1$. 6

8. (a) Prove that if two vectors of constant magnitudes undergo parallel displacement along a given curve, then they are inclined at a constant angle. 4

(b) Show that the geodeics are auto parallel curves. Determine a relation between metric tensors a_{ij} and g_{ij}. 3+2=5

(c) State and prove the fundamental theorem of Riemannian geometry. 5

UNIT–V

9. (a) Obtain an expression for Riemannian-Chrostoffel tensor of second kind. 8

(b) Prove that the curvature tensor of second kind can be contracted in two ways– one of these leads to zero and the other to a symmetric tensor. 6

10. (a) Define Ricci's coefficient of rotation. Obtain the necessary and sufficient conditions that a congruence be a geodesic congruence. 6

(b) Prove the necessary and sufficient conditions that n–1 congruences $e_{h|}$ of an orthogonal ennuple be canonical with respect to $e_{n|}$ are $\gamma_{nhk} = -\gamma_{nkh}$; $(h, k = 1, 2, ..., n-1 \text{ s.t. } h \neq k)$. 8