2021

M.Sc.

Third Semester

CORE – 10

MATHEMATICS

Course Code: MMAC 3.21

(Topology)

Total Mark: 70 *Pass Mark: 28*

Time: 3 hours

Answer five *questions, taking one form each unit.*

UNIT-1

1. (a) Let X be a topological space. Suppose $\mathbb{C}$ is a collection of open sets of X such that for each open set U of X and each $x \in U$ there is an element C of $\mathbb{C}$ such that $x \in C \subset U$. Then prove that $\mathbb{C}$ is a basis for the topology of X. 5

   (b) Show that the countable collection $B = \{(a,b) / a < b, a \text{ and } b \text{ are rational}\}$ is a basis that generates the standard topology on $\mathbb{R}$. 5

   (c) If $\mathfrak{B}$ is a basis for the topology of X and $\mathbb{C}$ is a basis for topology of Y, then prove that the collection $\mathfrak{D} = \{B \times C / B \in \mathfrak{B}, C \in \mathbb{C}\}$ is a basis for the topology on $X \times Y$. 4

2. (a) Let Y be a subspace of X. Prove that a set A is closed in Y if and only if it equals the intersection of a closed set in X with Y. 5

   (b) Let A be a subset of a topological space X. Prove that $x \in \overline{A}$ if and only if every open set U containing x intersects A. 4

   (c) Let A be a subset of the topological space X; let A' be the set of all limit points of A. Prove that $\overline{A} = A \cup A'$. 5

UNIT-II

3. (a) Let X and Y be topological spaces; let $f : X \to Y$. Prove that the following statements are equivalent:

(i) f is continuous

(ii) For every subset A of X, $f(\overline{A}) \subset \overline{f(A)}$

(iii) For every closed set B of Y, $f^{-1}(B)$ is closed in X. 6

(b) Define the following:

(i) Homeomorphism

(ii) Closed functions

(iii) Open functions

(iv) Topological property 4

(c) Suppose $f : X \to Y$ is continuous. If x is a limit point of the subset A of X, is it necessarily true that $f(x)$ is a limit point of $f(A)$? Justify. 4

4\. (a) Prove that for functions $f : \mathbb{R} \to \mathbb{R}$, the $\varepsilon - \delta$ definition of continuity implies the open set definition. 5

(b) Define topological imbedding. 4

(c) Define quotient map and quotient topology induced by a surjective map. 5

UNIT-III

5\. a) Prove that if a topological space is 2^{nd} countable, then it is separable and Lindelof. 6

(b) Prove that in a 1^{st} countable space, sequential continuity and continuity are equivalent. 4

(c) Prove that the continuous image of a Lindelof space is Lindelof. 4

6\. (a) Let X be a topological space. Prove that X is regular if and only if given a point $x \in X$ and an open set U of x, there is an open set V such that $x \in V \subset \overline{V} \subset U$. 6

(b) In a first countable space prove that X is T_2 if and only if every convergent sequence in X has a unique limit. 4

(c) Prove that a completely regular space is regular. 4

UNIT-IV

7\. (a) Prove the followings: (provide justification wherever it is necessary)

(i) closed subsets of compact space is compact

(ii) compactness is not an hereditary property 3+1=4

(b) Prove that a compact subset of a T_2 space is closed. 5

(c) Prove that continuous image of a compact space is compact. 5

8. (a) Prove that every compact Hausdroff space is normal. 6

(b) Let X be limit point compact. If A is closed subset of X, does it follow that A is limit point compact? Justify. 4

(c) Let X be locally compact and T_2. Let A be a subspace of X. If A is closed in X or open in X, then prove that A is locally compact.. 4

UNIT–V

9. (a) Prove that if X is having a separation and Y is a connected subset of X, then Y has to entirely lie in one of the open sets of the separation. 5

(b) Let $\{A_n\}$ be a sequence of connected subspaces of X such that $A_n \cap A_{n+1} \neq \phi$ for all n. Show that $\cup A_n$ is connected. 5

(c) Let A be a connected subspace of X. If $A \subset B \subset \overline{A}$, then prove that B is also connected. 4

10. (a) Prove that each non-empty connected subspace of X intersects only one connected component of X. 5

(b) Define a path between two points in a topological space. Prove that $x \sim y$ (x is related to y), if there is a path in X from x to y, is an equivalence relation. 5

(c) Prove that a space X is locally connected if and only if for every open set U of X, each component of U is open in X. 4