2021

M.Sc.

First Semester

CORE – 04

MATHEMATICS

Course Code: MMAC 1.41

(Abstract Algebra)

Total Mark: 70 *Pass Mark: 28*

Time: 3 hours

Answer five questions, taking one from each unit.

UNIT–I

1. (a) Prove that the set of all positive rational numbers form an abelian group under the operation defined by $a * b = \frac{ab}{2}$. 5

(b) Prove that the centre of a group is a subgroup of the group. 5

(c) Show that a homomorphism $f : G \to G'$ is one-one if and only if $\ker f = \{e\}$, where G and G' are groups. 4

2. (a) Prove that the union of the subgroups of a group is a subgroup if and only if one is contained in the other. 6

(b) Prove that a subgroup H of a group G is a normal subgroup of G if and only if every left coset of H in G is a right coset of H in G. 4

(c) Show that A_4 is the only subgroup of order 12 in S_4. 4

UNIT–II

3. (a) Explain group action by conjugation. 4

(b) State and prove Cayley's theorem. 6

(c) If G is a finite group, show that G is a p-group if and only if $O(G) = p^n$. 4

4. (a) State and prove Sylow's first theorem. 8

(b) Prove that the dihedral group D_n has $2n$ elements. 6

UNIT–III

5. (a) Show that no group of order 30 is simple. 5

(b) Show that the relation of conjugacy is an equivalence relation. 4

(c) Show that $G_1 \times G_2$ is abelian if and only if both G_1 and G_2 are abelian. 5

6. (a) If G is a group of order pqr, $p<q<r$, where p, q and r are distinct primes, show that G is not simple. 7

(b) Find all the conjugate classes in S_4 and verify the class equation. 7

UNIT–IV

7. (a) If A and B are two ideals of a ring R, prove that $A+B$ is an ideal of R, containing both A and B. 4

(b) Prove that a Euclidean domain is a PID. 6

(c) If $\mathbb{Z}$ is the ring of integers, show that the only homomorphisms from $\mathbb{Z}$ to $\mathbb{Z}$ are the identity and zero mappings. 4

8. (a) Prove that in a UFD an element is prime if and only if it is irreducible. 6

(b) If R and S are two rings, prove that

$$ch(R \times S) = \begin{cases} 0, & \text{if } ch\ R = 0 \text{ or } ch\ S = 0 \\ k, & \text{where } k = lcm(ch\ R, ch\ S), \text{ otherwise} \end{cases}$$ 4

(c) If R is a ring with unity such that R has no right ideals except $\{0\}$ and R, show that R is a division ring. 4

UNIT–V

9. (a) Prove that the product of two primitive polynomials is also a primitive polynomial. 5

(b) If F is a field, prove that $F[x]$ is a PID. 5

(c) For any prime p, show that the polynomial

$x^{p-1} + x^{p-2} + \ldots + x^2 + x + 1$ is irreducible over $\mathbb{Q}$. 4

10. (a) State and prove Eisenstein's criterion for irreducibility. 6

(b) Show that the polynomial $x^2 + x + 2$ is irreducible in $\mathbb{Z}_3[x]$ and use it to construct a field of 9 elements. 4

(c) Show that the polynomial $x^2 + 1$ is irreducible over $\mathbb{Z}_3$ but reducible over $\mathbb{Z}_5$. 4