2021

M.Sc.

First Semester

CORE – 03

MATHEMATICS

Course Code: MMAC 1.31

(Real Analysis)

Total Mark: 70 *Pass Mark: 28*

Time: 3 hours

Answer five *questions, taking* one *from each unit.*

UNIT–I

1. (a) If X is countable and $Y \subseteq X$, then prove that Y is at most countable. 5
   (b) Prove that the interior of a set A is the largest open set in A. 5
   (c) Is $\{1,2,3\} \times (\mathbb{R} \setminus \mathbb{N})$ countable? Justify. 4

2. (a) Prove that the Cartesian product of two countable sets is countable. 6
   (b) Let X be an infinite set and $x \in X$. Exhibit a bijection between X and $X \setminus \{x\}$. 4
   (c) Is $A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$ compact in $\mathbb{R}$? Justify. 4

UNIT–II

3. (a) Let (x_n) be a sequence in a metric space (X, d) and let $x \in X$. Prove that (x_n) converges to x if and only if for all open subset V of X containing x, $\exists N \in \mathbb{N}$ such that $x_n \in V, \forall n \geq N$. 7

(b) Prove that e is irrational. 7

4. (a) Let (X, d) be a metric space, $A \subseteq X$ and $x \in \overline{A}$. Prove that there exists a sequence in A which converges to x. 7

(b) Discuss the convergence of $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ 7

UNIT–III

5. (a) Let $f : X \to Y$ be continuous, where X, Y are metric spaces.

If F is a closed set in Y, prove that $f^{\leftarrow}(F)$ is closed in X.

If A is a closed set in X, is $f(A)$ closed in Y? Justify. 8

(b) If $f :]a, b[\to \mathbb{R}$ is differentiable and $f'(x) \geq 0$, $\forall x \in]a, b[$, then is f monotonically increasing or decreasing? Justify. 6

6. (a) Let $f : X \to Y$ be continuous, where X, Y are metric spaces. If X is connected, prove that $f(X)$ is also connected. 7

(b) Describe and justify the chain rule in differentiation. 7

UNIT–IV

7. (a) If $f : [a, b] \to \mathbb{R}$ is bounded and P, P^* are partitions of $[a, b]$ such that P^* is finer than P, then how are $L(P, f)$ and $L(P^*, f)$ related? 7

(b) If f is continuous in $[a, b] \subset \mathbb{R}$ and f is real-valued, prove that f is integrable. Is the converse true? 7

8. (a) Prove that $f:[a,b]\to\mathbb{R}$ is integrable if and only if given $\varepsilon>0$, there exists $\delta>0$, a partition $P=\{x_0,x_1,...,x_n\}$ of $[a,b]$ with $\|P\|<\delta$ and a real number L such that $\forall t_i\in[x_{i-1},x_i]$, we have $\left|\sum_{i=1}^{n} f(t_i)(x_i-x_{i-1})-L\right|<\varepsilon$. 8

(b) If $f:[a,b]\to\mathbb{R}$ is bounded and has a point of discontinuity, is f integrable? Justify. 6

UNIT–V

9. (a) For $x\in\mathbb{R}, x\geq 0$, evaluate the following. 6

(i) $\lim\left(\frac{\sin nx}{1+nx}\right)$

(ii) $\lim\left(e^{-nx}\right)$

(b) State and prove the Cauchy criterion for uniform convergence. 8

10. (a) Show that if $a>0$, then the convergence of the sequence $\left(\frac{x}{x+n}\right)$ is uniform on the interval $[0,a]$, but is not uniform on the interval $[0,\infty[$. 8

(b) Show that for all $x\in\mathbb{R}, x\geq 0$ 6

(i) $\lim\left(\frac{x}{x+n}\right)=0$

(ii) $\lim\left(xe^{-nx}\right)=0$