2021
M.Sc.
First Semester
CORE – 02
MATHEMATICS
Course Code: MMAC 1.21
(Linear Algebra)

Total Mark: 70 *Pass Mark: 28*
Time: 3 hours
Answer five questions taking one from each Unit.

UNIT–I

1. (a) Prove that a non-empty subset W of a vector space V over the field F is a subspace of V if and only if for each pair of vectors $u, v \in W$ and each scalar $\alpha \in F$, the vector $au + v \in W$. 7

(b) If $T \in \mathcal{L}(V, W)$, define $null\, T$ with an example. Prove that T is injective if and only if $null\, T = 0$. 7

2. (a) Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other. 7

(b) Suppose V is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that $\dim V = \dim null\, T + \dim range\, T$

UNIT–II

3. (a) Suppose V is finite dimensional and $T \in \mathcal{L}(V)$. Prove that the following are equivalent:
(i) T is invertible
(ii) T is injective
(iii) T is surjective 7

(b) If $T \in \mathcal{L}(V)$, prove that the zeros of the minimal polynomial of T are precisely the eigenvalues of T. 7

4. (a) Let $T \in \mathcal{L}(V)$. Suppose $\lambda_1, \lambda_2, ..., \lambda_m$ are distinct eigenvalues of T and $v_1, v_2, ..., v_m$ are corresponding eigen vectors. Prove that $v_1, v_2, ..., v_m$ are linearly independent. 6

(b) Suppose $S, T \in \mathcal{L}(V)$ and S is invertible. Prove that if $p \in \mathcal{P}(F)$ is a polynomial, then $p\left(STS^{-1}\right) = Sp(T)S^{-1}$. 4

(c) Suppose $T \in \mathcal{L}\left(\mathbb{C}^3\right)$ is defined by $T\left(z_1, z_2, z_3\right) = \left(6z_1 + 3z_2 + 4z_3, 6z_2 + 2z_3, 7z_3\right)$. Find the eigenvalues of T and the corresponding generalized eigenspaces of T. 4

UNIT–III

5. (a) Suppose $S, T \in \mathcal{L}(V)$ are such that $ST = TS$. Prove that $null\, S$ and $range\, S$ are invariant under T. 4

(b) If V is a finite-dimensional complex vector space and $T \in \mathcal{L}(V)$, then prove that T has an upper-triangular matrix with respect to some basis of V. 6

(c) If $T \in \mathcal{L}(V)$ and $\lambda \in F$, then prove that $G(\lambda, T) = null\,(T - \lambda I)^{\dim V}$. 4

6. (a) Suppose $N \in \mathcal{L}(V)$ is nilpotent, then prove that $I + N$ has a square root. 6

(b) If $N \in \mathcal{L}(V)$ is nilpotent, prove that there exist vectors $v_1, v_2, ..., v_n \in V$ and non-negative integers $m_1, m_2, ..., m_n$ such that

(i) $N^{m_1}\left(v_1\right), ..., N\left(v_1\right), v_1, ..., N^{m_n}\left(v_n\right), ..., N\left(v_n\right), v_n$ is a basis of V

(ii) $N^{m_1+1}(v_1) = \ldots = N^{m_n+1}(v_n) = 0$. 8

UNIT–IV

7. (a) State and prove the Cauchy-Schwarz inequality. 4

(b) Find an orthonormal basis of $\mathcal{P}_2(\mathbb{R})$, where the inner product is given by $\langle p,q \rangle = \int_{-1}^{1} p(x)q(x)\,dx$. 6

(c) If U is a finite-dimensional subspace of a vector space V, then prove that $V = U \oplus U^{\perp}$. 4

8. (a) Explain the Gram-Schmidt procedure. 7

(b) Find $u \in \mathcal{P}_2(\mathbb{R})$ such that $\int_{-1}^{1} p(t)\cos(\pi t)\,dt = \int_{-1}^{1} p(t)u(t)\,dt$ for every $p \in \mathcal{P}_2(\mathbb{R})$. 4

(c) Define $T : \mathbb{R}^3 \to \mathbb{R}^2$ by $T(x_1, x_2, x_3) = (x_2 + 3x_3, 2x_1)$. Find a formula for T^*. 3

UNIT–V

9. (a) Define a map $B : \mathcal{P}_3(\mathbb{R}) \times \mathcal{P}_3(\mathbb{R}) \to \mathbb{R}$ given by

$$B(p(x), q(x)) = \int_{0}^{1} p(t)q(t)\,dt$$. Show that B is bilinear.

Determine the matrix of B relative to ordered basis $1, x, x^2, x^3$ of $\mathcal{P}_3(\mathbb{R})$. 6

(b) Prove that every eigenvalue of a self-adjoint operator is real. 4

(c) Let V be a vector space of dimension n over F and $Bil(V)$ be the set of all bilinear forms on V. Define addition and scalar multiplication in $Bil(V)$ as follows.

$$(B_1 + B_2)(x, y) = B_1(x, y) + B_2(x, y)$$

$$(\alpha B_1)(x, y) = \alpha B_1(x, y)$$

for all $B_1, B_2 \in Bil(V), \alpha \in F, x, y \in V$. Prove that $Bil(V)$ is a vector space of dimension n^2 over F. 4

10. (a) An operator $T \in \mathcal{L}(V)$ is normal if and only if $\|T(v)\| = \|T^*(v)\|$. 5

(b) If B is a symmetric bilinear form on a finite-dimensional vector space V over F and if $e_1, e_2, ..., e_n$ is any orthonormal basis of V, then prove that the number of e_i's such that $B(e_i, e_i) = 0$ is equal to the dimension of $V^{\perp}$. 5

(c) Let V be the vector space of all continuous complex-valued functions on the interval $-\pi \le x \le \pi$. Define $H : V \times V \to \mathbb{C}$ by $H(f, g) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\overline{g(t)}\,dt$ for all $f, g \in V$. Show that H is a positive definite Hermitian form. 4