2021

M.Sc.

First Semester

CORE – 01

MATHEMATICS

Course Code: MMAC 1.11

(Ordinary Differential Equations)

Total Mark: 70 *Pass Mark: 28*

Time: 3 hours

Answer five *questions, taking* one *from each unit.*

UNIT–I

1. (a) Consider the equation $Ly''+Ry'+\frac{1}{C}y=0$, where L, R and C are positive.

   (i) Compute all the solution $\frac{R^2}{L^2}-\frac{4}{LC}<0$ 4

   (ii) Show that any solution ϕ in case of (i) may be written in the form $\phi(x)=Ae^{\alpha x}\cos(\beta x-\omega)$, where A,α,β,ω are constants. Also determine α, β. 4

   (b) Find a function ϕ which has a continuous derivative on $0\le x\le 2$ which satisfies $\phi(0)=0, \phi'(0)=1$ where $y''-y=0$ for $0\le x\le 1$ and $y''-9y=0$ for $1\le x\le 2$. 6

2. (a) Let ϕ_n be any function satisfying the following boundary value problem:

   $y''+n^2y=0,\ y(0)=y(2\pi),\ y'(0)=y'(2\pi)$, where $n=0,1,2,...$

   Show that $\int_0^{2\pi}\phi_n(x)\phi_m(x)\,dx=0$ if $n\ne m$. 4

   (b) Find all the solutions of the following equations: 5×2=10

   (i) $y''+4y=\cos x$

(ii) $y''+y=\tan x,\left(-\frac{\pi}{2}<x<\frac{\pi}{2}\right)$.

UNIT–II

3. (a) Prove that there exist n linearly independent solutions of $L(y)=0$, on any interval I. 4

(b) Find two linearly independent solutions of the equation

$$(3x-1)^2 y^n+(9x-3)y'-9y=0 \text{ for } x>\frac{1}{3}.$$ 4

(c) One solution of $L(y)=y''+\frac{1}{4x^2}y=0$ for $x>0$ is $\phi(x)=x^{\frac{1}{2}}$.

Show that there is another solution ψ of the form $\psi=u\phi$ where u is some function. 6

4. (a) Let $\phi_1, \phi_2, \ldots, \phi_n$ be n linearly independent solutions of $L(y)=0$ on the interval I. If ϕ is any solution of $L(y)=0$ on the interval I, then prove that it can be represented in the form $\phi=c_1\phi_1+c_2\phi_2+\ldots+c_n\phi_n$, where $c_1, c_2, \ldots, c_n$ are constants. Thus any set of n linearly independent solutions of $L(y)=0$ on I is a basis for the solutions of $L(y)=0$ on I. 4

(b) Consider the equation $y''+\alpha(x)y=0$ where α is a constant function on $-\infty<x<\infty$ which is of period $\xi>0$. Let ϕ_1, ϕ_2 be the basis for the solutions satisfying

$\phi_1(0)=1,\ \phi_2(0)=0;\ \phi_1'(0)=0;\ \phi_2'(0)=1$.

(i) Show that $W(\phi_1,\phi_2)(x)=1$ for all x. 2

(ii) Show that there exist a non trivial solution ϕ of period ξ if and only if $\phi_1(\xi)+\phi_2'(\xi)=2$. 4

(iii) Show that there exists a non-trivial solution ϕ satisfying

$\phi(x+\xi) = -\phi(x)$ if and only if $\phi_1(\xi) + \phi_2'(\xi) = -2$. 4

UNIT–III

5. (a) Show that there is a basis ϕ_1, ϕ_2 for the solution of

$x^2 y'' + 4xy' + (2 + x^2) y = 0$, $(x > 0)$ of the form

$\phi_1(x) = \dfrac{\psi_1(x)}{x^2}, \phi_2(x) = \dfrac{\psi_2(x)}{x^2}$. Also find the solutions of

$x^2 y'' + 4xy' + (2 + x^2) y = x^2$ for $x > 0$ 5+5=10

(b) The equation $y'' + e^x y = 0$ has a solution of the form

$\phi(x) = \sum_{k=0}^{\infty} c_k x^k$, which satisfies $\phi(0) = 1,\ \phi'(0) = 1$

. Compute $c_0, c_1, c_2, c_3, c_4, c_5$ 4

6. (a) Find two linearly independent power series solutions of

(i) $y'' - xy' + y = 0$ 5

(ii) $y'' + 3x^2 y' - xy = 0$ 5

(b) If P_n is n^{th} Legendre polynomial, then show that

$P_n(-x) = (-1)^n P_n(x)$ 4

UNIT–IV

7. (a) Solve the equation $y' = \dfrac{1}{2}\left(\dfrac{x + y - 1}{x + 2}\right)^2$ 4

(b) Find the integrating factor and solve the equation

$\cos x \cos y \, dx - 2 \sin x \sin y \, dy = 0$. 4

(c) If the equation $M\, dx + N\, dy = 0$ has an integrating factor u which is

a function of x alone, and $u\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right) = N\dfrac{\partial u}{\partial x} - M\dfrac{\partial u}{\partial y}$, then

(i) show that $p=\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)$ is a function of x alone.

(ii) if is p continuous and independent of y, show that an integrating factor is given by $u(x)=e^{P(x)}$, where P is any function satisfying $P'=p$. 6

8. (a) For each of the following problems, compute the first four successive approximations: 6

(i) $y'=x^2+y^2,\ y(0)=0$,

(ii) $y'=1+xy,\ y(0)=1$

(b) Consider the problem $y'=1+y^2,\ y(0)=0$,

(i) Using separation of variables, find the solution of this problem. 2

(ii) Show that all the successive approximations $\phi_0,\phi_1,\phi_2,...$ exist for all real x. 4

(iii) Show that $\phi_k(x)\to\phi(x)$ for each satisfying $|x|\le\frac{1}{2}$. 2

UNIT–V

9. (a) Find all the solutions of the equation: $x^2y''+xy'-4y=x,\ (x>0)$. 7

(b) Find the singular point and compute the indicial polynomial and the roots of the equation $x^2y''+(x+x^2)y'-y=0$ 7

10. (a) Show that $x=0$ is a regular singular point of the equation $x^2y''+xy'+\left(x^2-n^2\right)y=0$ and find its solution taking $2n$ as non integer. 10

b) If $J_n(x)$ denotes the Bessel's function of 1st kind, prove that

$$\frac{d}{dx}\left\{x^nJ_n(x)\right\}=x^nJ_{n-1}(x)$$ 4