2021

B.A./B.Sc.

Fifth Semester

CORE – 12

MATHEMATICS

Course Code: MAC 5.21

(Group Theory - II)

Total Mark: 70 *Pass Mark: 28*

Time: 3 hours

Answer five *questions, taking* one *from each unit.*

UNIT–I

1. (a) Define automorphism of a group. Prove that the set of all automorphisms of a group forms a group with respect to composite of functions as the composition. 6

   (b) If G is an infinite cyclic group, determine the group of automorphisms of G. 4

   (c) If G is a group, f an automorphism of G and N a normal subgroup of G, prove that $f(N)$ is a normal subgroup of G. 4

2. (a) Define inner automorphism and prove that the set of all inner automorphisms of a group is a normal subgroup of the group of its automorphism. 5

   (b) If G' is the commutator subgroup of a group G, prove that G' is normal in G and G/G' is abelian. 5

   (c) Define characteristic subgroup of a group. Show that a characteristic subgroup of a group G is a normal subgroup of G. Is the converse true? 4

UNIT–II

3. (a) Show that the external direct product $G_1 \times G_2$ is a group for the binary operation defined by $(g_1, g_2)(h_1, h_2) = (g_1h_1, g_2h_2)$, where $g_1, h_1 \in G_1$ and $g_2, h_2 \in G_2$. 5

(b) Find the number of elements of order 5 in $Z_{25} \oplus Z_5$. 4

(c) Show that a group of order 4 is either cyclic or is an internal direct product of two cyclic groups of order 2 each. 5

4. (a) If G_1 and G_2 are any two groups, show that $G_1 \times G_2$ is abelian if and only if both G_1 and G_2 are abelian. 5

(b) If G is an internal direct product of its subgroups H and K, prove that $H \cap K = \{e\}$ and that G is isomorphic to the external direct product of H and K. 5

(c) Find all the non-isomorphic abelian groups of order 360. 4

UNIT–III

5. (a) Define group action and explain group action by conjugation. 4

(b) Define kernel of a group action and prove that it is a subgroup of the group. 5

(c) State and prove the index theorem. 5

6. (a) If G is any group and A is a non-empty set, prove that any homomorphism from G to S_A, the symmetric group of A defines an action of G on A. Also, prove the converse. 6

(b) If G is a group and $a \in G$, define stabilizer of a in G and prove that it is a subgroup of G. 4

(c) If G is a group and A is any non-empty set, prove that $* : G \times A \to A$ defined by $g * a = a, \forall a \in A, g \in G$, is a group action. 4

UNIT–IV

7. (a) Let G be a group and $\sim$ the relation in G given by $b \sim a$ if and only if b is conjugate to a. Show that $\sim$ is an equivalence relation. 5

(b) Find all the conjugate classes in S_3 and verify the class equation. 5

(c) Obtain the class equation for a finite group. 4

8. (a) If G is a group, prove that the number of elements in the conjugate class $cl(a)$, where $a \in G$, is equal to the index of the normalizer $N(a)$

of a in G. 5

(b) Find all the conjugate classes in S_4 and verify the class equation. 5

(c) If G is a group and $a \in G$, prove that $cl(a) = \{a\}$ if and only if $a \in Z(G)$, where $Z(G)$ is the centre of G. 4

UNIT–V

9. (a) State and prove Cauchy's theorem for finite groups. 6

(b) Prove that there are no simple groups of order 63. 4

(c) If the order of a group G is p^2q^2, where p and q are distinct prime, show that G is not simple. 4

10. (a) Prove that any group of order 15 is cyclic, using Sylow's theorem. 5

(b) Prove that the alternating group A_5 is simple. 6

(c) Prove that a Sylow p-subgroup H of a group G is normal if H is the unique sylow p-subgroup of G. 3
