2021

B.A./B.Sc.

Third Semester

CORE – 5

MATHEMATICS

Course Code: MAC 3.11

(Theory of Real Functions)

PART–B

Total Mark: 30

Answer the following questions. 6×5=30

1. (a) Use the definition of limit to show that $\lim\limits_{x\to 1}\frac{x+5}{2x+3}=\frac{6}{5}$. 2

   (b) State and prove squeeze theorem. 2

   (c) Give examples of functions *f* and *g* such that *f* and *g* do not have limits at a point *c* but such that both $f+g$ and fg have limits at *c*. 2

2. (a) Let $f:\mathbb{R}\to\mathbb{R}$ be continuous on $\mathbb{R}$ and let $S=\{x\in\mathbb{R}\ / f(x)=0\}$. If $\{x_n\}$ is a sequence in S and $x=\lim\limits_{n\to\infty}x_n$, show that $x\in S$. 2

   (b) Let $I=[a,b]$ be closed and bounded interval and let $f:I\to\mathbb{R}$ be continuous on *I*. Prove that *f* has an absolute maximum and an absolute minimum on *I*. 2

   (c) Prove that the function $f(x)=2x-8$ is uniformly continuous and also satisfies the Lipschitz condition. Give an example of a function that is uniformly continuous but does not satisfy the Lipschitz condition. 2

3. (a) Assume that there exists a function $L:(0,\infty)\to\mathbb{R}$ such that $L'(x)=\frac{1}{x}$ for $x>0$. Calculate the derivative of the following functions.

   (i) $f(x)=L(2x+3)$ for $x>0$

   (ii) $g(x)=\left(L\left(x^2\right)\right)^3$ for $x>0$

   (iii) $h(x)=L(ax)$ for $a>0,\ x>0$

   (iv) $k(x)=L(L(x))$ when $L(x)>0,\ x>0$ 2

   (b) State and prove intermediate value property of derivatives. 2

   (c) Use the mean value theorem to prove that $1-\frac{1}{x}<\log x<x-1$ for $x>1$. 2

4. (a) Evaluate $\lim\limits_{x\to 0}\frac{x^2-\sin^2 x}{x^4}$ 2

(b) Evaluate $\lim_{x \to 0^+} \frac{\log \cos x}{x}$ in $\left(0, \frac{\pi}{2}\right)$ 2

(c) Evaluate $\lim_{x \to 0} \left(1 + \frac{3}{x}\right)^x$ in $(0, \infty)$ 2

5. (a) State and prove Taylor's theorem with Lagrangian form of remainder. 2

(b) Show that if $x > 0$, then $1 + \frac{1}{2}x - \frac{1}{8}x^2 \leq \sqrt{1+x} \leq 1 + \frac{1}{2}x$. 2

(c) Determine whether or not $x = 0$ is a point of relative extremum for the following functions:

(i) $f(x) = x^2 + 2$

(ii) $g(x) = \sin x + \frac{1}{6}x^3$

(iii) $h(x) = \cos x - 1 + \frac{x^2}{2}$

(iv) $k(x) = \sin x - x$ 2